LOOP-ERASED RANDOM WALK ON A TORUS IN DIMENSIONS 4 AND ABOVE

ITAI BENJAMINI AND GADY KOZMA

1. INTRODUCTION

A well known phenomenon in probabilistic constructions in $\mathbb{R}^d$ or $\mathbb{Z}^d$ is that usually some critical dimension d exists, above which the geometry of $\mathbb{R}^d$ ceases to play any significant role, and the process behaves like a similar non-geometric object, such as a tree, a complete graph, etc. Usually, this also corresponds to "mean field behavior", a term meaning that for the random variables of interest one has $\mathbb{E}X^n \approx (\mathbb{E}X)^n$. At the critical dimension itself, mean field behavior is also expected, but when compared to the non-geometric object one gets a logarithmic correction.

Many results confirming this general philosophy exist. See [HS90] for results about percolation, [HS92] for results about the self-avoiding walk, [DS98] for results about lattice trees, and [S95] for a general survey. In particular, the problem of loop-erased random walk on $\mathbb{Z}^d$ is well studied.

Loop-erased random walk is a process that starts from a random walk on some graph and then removes all loops in chronological order, or in other words, whenever the random walk hits the partial path, the loop just created is erased and the process continues. The result is a random simple path. Originally [L80] suggested as a model for the self-avoiding walk (a random walk conditioned not to hit itself), better understanding of its structure has situated it as an important object in combinatorics and mathematical physics. See [S00] for a survey, and the complementary [LSW]. For a survey with a different focus, see [L99]. Note also the recent [BKPS] — the uniform spanning tree is an object closely related to loop-erased random walk, but the structure of its phase transitions in various dimensions is richer. For other recent results of interest, see [BLPS01, K, LPS].

It is well known that the critical dimension of loop-erased random walk on $\mathbb{Z}^d$ is 4, since above this dimension a random walk does not intersect itself enough and the process of loop-erasure is local and uninteresting. See [L96, chapter 7]. Further, loop-erased walk is one of the few models where the logarithmic correction is known precisely, with a correction of $\log^{-1/3}$ loop-erased random walk on $\mathbb{Z}^4$ is similar to the regular random walk on $\mathbb{Z}^4$, see [L95].

With so much known, it seems strange that a small change in settings could provoke significant difficulties. To understand why, let us examine the question we are interested in precisely. Let T be a discrete torus, $\mathbb{Z}^d/(N\mathbb{Z})^d$ for some large N. Let b and e be two points on a torus, and let R be a random walk starting from b and stopped on e. We wish to say something about the loop-erasure of R. The results for $\mathbb{Z}^d$ all use the fact that the random walk does not intersect itself enough. However, in our settings the random walk does a very long walk — of the order of N^d — in a relatively small space, and intersects itself over and over again. Thus

it is definitely **not** true that the random walk and its loop erasure are similar! The random walk is essentially a random set that covers a large portion of the torus. Its loop-erasure is much thinner — as we will see, the expected size is $N^{d/2}$.

The geometry-less model we have in mind is the complete graph. There are a number of ways this model can be analyzed, but our favorite is using the notion of the Laplacian random walk. A Laplacian random walk from b to e, two points on an arbitrary graph G, is constructed inductively by solving, at each step, the discrete Dirichlet problem

$$f(e) = 1, \ f|_\gamma \equiv 0, \ \Delta f|_{G \backslash (\gamma \cup \{e\})} \equiv 0 \tag{1}$$

where γ is the partially constructed path and Δ is the discrete Laplacian. The walk then continues to the next point using f as weights. This model was suggested in [LEP86] and was shown to be equivalent to loop-erased random walk in [L87]. The case of the complete graph is very easy to analyze, since if the partially constructed curve γ has length i then

$$f(v) = \begin{cases} 0 & v \in \gamma \\ 1 & v = e \\ \frac{1}{i+1} & \text{otherwise} \end{cases} .$$

and then the probability of the walk to terminate in the next step is $\frac{i}{N}$. This gives a closed formula

$$\mathbb{P}(\# \operatorname{LE}(R) = k) = \frac{k-1}{N} \prod_{i=1}^{k-2} \left(1 - \frac{i}{N}\right) .$$

In particular, we see that the correct scaling is $\sqrt{N}$ and that $\# \operatorname{LE}(R)/\sqrt{N}$ converges to a limiting distribution with density $te^{-t^2/2}$. Unfortunately, we do not know how to analyze more interesting graphs using the Laplacian random walk, nor can we show the existence of a limiting distribution for $\# \operatorname{LE}(R)$ on, say, the torus.

Thus we have a good basis to claim that mean field behavior in our case should be $\sqrt{|T|} = N^{d/2}$. For $d < 4$ this does not happen — indeed known results for $d = 1$ (trivial) and $d = 2$ ([K00a, K00b], see also [LSW]) and computer simulations for $d = 3$ [GB90] show that even a single branch of the loop-erased walk is too big[1]. We shall show that mean field behavior does occur for $d > 4$. In the critical dimension itself, we can only show an upper bound, and we do not calculate the precise logarithmic correction (we do have some good evidence for a conjecture on the precise logarithmic correction needed — $\log^{1/6} N$ — see page 12). Namely, our results are

Theorem 1. *If $d > 4$ then a loop-erased random walk L on the (N, d)-torus starting from a point b and stopped when hitting a point e has the estimate*

$$\mathbb{P}(\#L > \lambda N^{d/2}) \le Ce^{-c\lambda} .$$

If $d = 4$ then

$$\mathbb{P}(\#L > \lambda N^{2+\epsilon}) \le Ce^{-c\lambda} \quad \forall \epsilon > 0.$$

Where the constants C and c may depend on d and on ϵ.

---

[1]We believe that the growth exponents in $d = 2, 3$ are the same on $\mathbb{Z}^d$ and T_N^d, but this is beyond the scope of this paper.

Theorem 2. *Let $d \geq 5$. Let b be a point in $T = T_N^d$ and let e be a random, uniform point in T. Let R be a random walk on T starting from b and stopped at e. Let $\lambda \geq N^{-1/2}$. Then*

$$\mathbb{P}(\# \operatorname{LE}(R) \leq \lambda N^{d/2}) \leq C\lambda \log \lambda^{-1}$$

Returning to the cases of $d \leq 3$, we see that the reason for non-mean-field behavior is strong local intersections and these *increase* the size of the loop-erased walk. Therefore we are tempted to conjecture

Conjecture. *Let G be a vertex transitive finite graph, and let b and e be two random points in G. Let R be a random walk starting from b and stopped when hitting e. Then*

$$\mathbb{E}\# \operatorname{LE}(R) \geq c\sqrt{|G|}$$

A graph G is vertex transitive when, for every two vertices v and w there exists a graph automorphism of G carrying v to w. The requirement that G is vertex transitive is supported by the standard "extreme non-transitive" example of a tree of size N, where the loop-erased random walk between b and e is of course the *only* path between b and e and its length is bounded by $C \log N$.

We wish to end this introduction with one last conjecture. Returning to the analysis of the complete graph using the Laplacian random walk, we note that this analysis does not change by much if one considers α-power Laplacian random walk, which is a walk one gets if one takes as weights for any step the function f^α where f is defined by (1) — this generalization was also discussed in [LEP86]. For the complete graph we get that the size of a typical path is $N^{1/(1+\alpha)}$. We ask: is this behavior replicated in a d-dimensional torus for $d > d_\alpha^{\text{crit}}$?

Conjecture. *For $\alpha \leq 1$, $d_\alpha^{\text{crit}} = \frac{2(1+\alpha)}{\alpha}$, i.e. for any $d > \frac{2(1+\alpha)}{\alpha}$ the typical path of a α-weighted Laplacian random walk on a d-dimensional torus is of size $N^{d/(1+\alpha)}$ while for smaller d's this does not hold.*

We have no good conjecture on the value of the critical dimension for $\alpha > 1$, though it does seem (again, we have no proof of that) that for $\alpha = \infty$ (which corresponds to a non-probabilistic process which simply proceeds to the point where f attains its maximum) the process gives a straight line from b to e in all dimensions, so one might say the critical dimension is 1.

We wish to thank Chris Hoffman, Dan Romik and Oded Schramm for useful discussions.

1.1. **About the proof.** The basic question behind the solution is "what is the probability of a random walk of length L will hit a loop-erased walk of length L?" (in dimension 4 we need to differentiate between these two lengths, but only by a sub-polynomial factor). When the probability is larger then some constant $c > 0$, then this is the L we seek, as this means that the probability of a loop-erased random walk to go further than λL is exponentially small in λ. Since a loop-erased walk is a complicated object, let us first ask "what is the probability of a random walk of length L will hit some set Ω of size L?" This probability is largest when Ω is rather spread out. Take as an example Ω to be a random collection of points on the torus. It is easy to calculate the expected number of intersections of a random walk with Ω and the second moment and to derive from both the estimate that the probability is $\approx L^2 N^{-d}$ so this gives that the L we look for is $< C N^{d/2}$. When the set Ω is rather dense — for example, for a ball — a similar calculation will give that

$L < CN^{d/2+1}$. The difference between Ω dense and sparse manifests itself in the calculation of the second moment — see (11)-(12) below.

However, Ω is not a general set but a loop-erased random walk. The arguments we sketched above can be done locally, and we'd get that in every ball of radius r, the loop-erased random walk is not much larger than $r^{d/2+1}$. Effectively, this means that the set is spread out. We take this estimate and plug it directly into the calculation of the second moment and get a much better estimate for the intersection probability. Thus the proof is *recursive*, getting better estimates at each step. For $d > 4$ two or at most three steps are necessary to get the true estimate, $N^{d/2}$. This argument is done in lemma 1.

1.2. **Reading recommendations.** Section 2 is probably the one deserving most attention. While the main ideas are sketched above, the devil is in the details and the interested reader might want to read through the proof and do the "exercise" — not so designated explicitly — of simplifying the proof with a cost of $\sqrt{\log}$ in the final result. Section 3 is technical and most readers would probably agree that the conclusion (theorem 3) is not surprising. The proof of lemma 5 is the core — as for lemma 4, you might opt to read its statement but skip its proof. And again, verify that the claim is trivial if one is willing to lose a factor of $\log$ (the argument is contained in the first half-page of the proof of lemma 5). Section 4 contains the proof of theorem 2 and is quite short. While there are alternative, more complicated approaches that might prove a little more we have not included them. There are some comments and hints at the end of section 4 — we hope they make at least some sense. We have collected some well known and unsurprising facts we use (and their proofs) in the appendix. We hope this makes the paper more accessible to non-experts and students. Lemmas with numbers like "A.7" are to be found in the appendix.

1.3. **Standard notations.** In the sequel we denote by C and c positive constants which may depend on the dimension but on nothing else. C will usually pertain to constants which are "large enough" and c to constants which are "small enough". The notation $x \approx y$ is a short hand for $cx \leq y \leq Cx$. In dimension 4 we shall prove only imprecise estimates, namely that the length of the loop-erased walk is $< N^{2+\epsilon}$. All constants C and c may depend on this ϵ as well. Similarly, all constants implicit in notations such as O and $\approx$ might depend on d and ϵ. Occasionally we shall number constants for clarity. When we write $\log x$ we always mean $\max\{\log x, 1\}$ and $\log 0 = 1$.

The (N, d)-torus, denoted by T_N^d is the set $\mathbb{Z}^d/(N\mathbb{Z})^d$ endowed with the graph structure derived from $\mathbb{Z}^d$ and the distance derived from the l_2 norm on $\mathbb{Z}^d$. The distance of v and w will be denoted by $|v - w|$ while distance of sets will be denoted by $d(\cdot, \cdot)$. A ball of radius r and center v in either $\mathbb{Z}^d$ or T_N^d will be denoted by $B(v, r)$ and its inner boundary (namely, all points in B with an edge leading outside of B) by $\partial B(v, r)$.

2. THE UPPER BOUND

We will need to examine the effect of adding a section to a path and how it might increase the length of its loop-erasure. We shall always assume that the section we add starts at 0, so that we are looking at a path $\gamma : \{-m, \ldots, n\} \to T$ and define, in addition to the usual loop-erasure of γ, which we will denote by

$\mathrm{LE}(\gamma)$, the **continued** loop-erasure, which we shall denote by $\mathrm{LE}^+(\gamma)$. Here are both definitions:

Definition. For a finite path $\gamma : \{-m, \dots, n\} \to T$ in a graph T we define its loop erasure, $\mathrm{LE}(\gamma)$, which is a simple path in T, by the consecutive removal of loops from γ. Formally,

$$
\begin{aligned}
\mathrm{LE}(\gamma)_0 &:= \gamma(-m) \\
\mathrm{LE}(\gamma)_{i+1} &:= \gamma(j_i + 1) \quad j_i := \max\{j : \gamma(j) = \mathrm{LE}(\gamma)_i\} \quad .
\end{aligned}
$$

Naturally, this is defined for all i such that $j_i < n$. The continued loop-erasure is a subset of $\mathrm{LE}(\gamma)$ defined by

$$
\mathrm{LE}^+(\gamma)_i := \mathrm{LE}(\gamma)_{I+i} \quad I := \min\{i : j_i \geq 0\}
$$

The notations $\mathrm{LE}(\gamma[A, B])$ and $\mathrm{LE}^+(\gamma[A, B])$ stand for the loop-erasure and continued loop-erasure of the segment of γ going from A to B. When we write $-\infty$ in place of A we just mean the beginning of the path, nothing more.

Definition. Let $d \geq 4$ be the dimension and let $N \in \mathbb{N}$. Let R be a path in $T = T_N^d$ such that the negative part is fixed and the positive part is a random walk on T. Let $b = R(0)$. Let $v \in T$ and $0 < r < \frac{1}{8}N$, and assume for simplicity that $b \notin B(v, 2r)$. Let t_i be stopping times defined by $t_0 = 0$ and then inductively

$$
\begin{aligned}
t_{2i+1} &:= \min\{t \geq t_{2i} : R(t) \in \partial B(v, 2r)\} \tag{2} \\
t_{2i} &:= \min\{t \geq t_{2i-1} : R(t) \in \partial B(v, 4r)\} \quad .
\end{aligned}
$$

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Then we say that the (d-dimensional) random walk has the f**-property** if one has

$$
\mathbb{P}(\#(\mathrm{LE}^+(R[-\infty, t_i]) \cap B(v, r)) > \lambda f(r) \mid R[t_{2j}, t_{2j+1}] \forall j) \leq C e^{-c\lambda} \tag{3}
$$

which should hold for every such v and r, every $\lambda > 0$, every $i \in \mathbb{N}$ and any path we put in the negative portion of R.

The conditioning here, in words, is on any arbitrary set of paths between t_{2j} and t_{2j+1}, and in particular on the points $R(t_i)$ themselves. Notice that we do not condition on the value of the t_i's.

Let us remark that for the proof of the upper bound it is enough to consider the case where R has no negative part, and then $\mathrm{LE}^+ \equiv \mathrm{LE}$.

Lemma 1. *Let $d \geq 4$. Then*
(1) *If the d-dimensional random walk satisfies the $r^\alpha \log^\beta r$-property for $r^\alpha \log^\beta r \gg r^{d-2} \log^{-3} r$ then it also satisfies the $r^{\alpha/2+1} \log^{(\beta+3)/2} r$-property.*
(2) *If the d-dimensional random walk satisfies the $r^{d-2} \log^{-3} r$-property then it also satisfies the $r^{d/2} \sqrt{\log \log r}$-property.*
(3) *If it satisfies the $r^\alpha \log^\beta$-property for $r^\alpha \log^\beta r \ll r^{d-2} \log^{-3} r$ then it satisfies the $r^{d/2}$-property.*

Case 2 is not really necessary for the proof of the theorem, we include it here mainly for completeness.

Proof. Denote the function given to us (e.g. $r^\alpha \log^\beta r$) by $f(r)$ and the result (e.g. $r^{\alpha/2+1} \log^{(\beta+3)/2} r$) by $g(r)$. Let t_i be the stopping times from the definition of the f-property. The main part of the lemma will consider the events in $R[t_i, t_{i+1}]$ for

some particular odd i. Therefore let us fix $i > 0$. Denote $L_{i,v,r} := \#(\mathrm{LE}^+(R[0, t_i]) \cap B(v, r))$. Clearly $L_{2i+1,v,r} \le L_{2i,v,r}$ so if we prove the lemma for all i odd it will also hold for i even. To fix notations, we consider the time span $[-\infty, t_i]$ as the "past" and $]t_i, t_{i+1}]$ is the "present".

We start by examining the past. Let $w \in B(v, r)$ and $s \le \frac{1}{8}r$. The first step is to show that (3) holds if we replace the ball but keep the stopping times, i.e

$$\mathbb{P}(\#(\mathrm{LE}^+(R[0, t_i]) \cap B(w, s)) > \lambda f(s) \,|\, R[t_{2j}, t_{2j+1}]\forall j) \le Ce^{-c\lambda} \quad . \qquad (4)$$

We generalize the notation $L_{i,v,r}$ to $L_{i,w,s} := \#(\mathrm{LE}^+(R[0, t_i]) \cap B(w, s))$, that is, again, the loop-erased random walk inside a smaller ball measured at the stopping times pertaining to the larger ball.

Here our conditioning by everything outside the ball is crucial. Let $K_j \in \mathbb{N}$ be some arbitrary numbers, and let $\gamma_{j,k}$ be paths ($1 \le k \le K_j$) in $B(v, 4r) \setminus B(w, 2s)$ such that $\gamma_{j,1}$ is a path going from $R(t_{2j-1}) \in \partial B(v, 2r)$ to $\partial B(w, 2s)$, $\gamma_{j,k}$ for $1 < k < K_j$ is a path from $\partial B(w, 4s)$ to $\partial B(w, 2s)$ and γ_{j,K_j} is a path from $\partial B(w, 4s)$ to $R(t_{2j}) \in \partial B(v, 4r)$. If $K_j = 1$ then let $\gamma_{j,1}$ be a path from $R(t_{2j-1})$ to $R(t_{2j})$. Then we can sum over all such combinations of K and γ as follows. Denote by X the event $L_{i,w,s} > \lambda f(s)$. Let $Y_{K,\gamma}$ be the event that for all j, the random walk on $[t_{2j-1}, t_{2j}]$ follows $\gamma_{j,1}$ until $\partial B(w, 2s)$, then stays within $B(w, 4s)$, then follows $\gamma_{j,2}$ etc. until finally exiting from $B(v, 4r)$. Then

$$\mathbb{P}(X \,|\, R[t_{2j}, t_{2j+1}]\forall j) = \sum_{K,\gamma} \mathbb{P}(X \,|\, R[t_{2j}, t_{2j+1}]\forall j \cap Y_{K,\gamma}) \cdot \mathbb{P}(Y_{K,\gamma} \,|\, R[t_{2j}, t_{2j+1}]\forall j)$$

$$\le Ce^{-c\lambda} \sum_{K,\gamma} \mathbb{P}(Y_{K,\gamma} \,|\, R[t_{2j}, t_{2j+1}]\forall j) = Ce^{-c\lambda} \quad . \qquad (5)$$

Of course, we used the f-property for w, s and the index $\sum_{j=1}^{(i-1)/2} K_j$; and the fact that $B(w, 4s) \subset B(v, 2r)$.

The inequality (4) is not useful as it should be since most balls of radius s (for $s \ll r$) are empty anyway. However, another consequence of the conditioning is the fact that (4) is independent from the event $\mathrm{LE}^+(R[-\infty, t_i]) \cap B(w, 4s) = \emptyset$. The reason is that $L_{i,w,4s} = 0$ if and only if the segment inside $B(w, 4s)$ is cut "from the root", i.e. for some $u_1 < u_2 < \cdots < u_{2n}$, $n \in \{1, 2, \dots\}$ we must have $R[u_{2i-1}, u_{2i}] \cap B(w, 4s) = \emptyset$ and $R(u_{2i}) = R(u_{2i+1})$. Whether this happens in the positive or negative part of R is immaterial — in both cases this is an event that happens outside $B(w, 4s)$ therefore it is an event we condition on. We get

$$\mathbb{P}(L_{i,w,s} > \lambda f(s) \,|\, R[t_{2j}, t_{2j+1}]\forall j) \le Ce^{-c\lambda}\mathbb{P}(L_{i,w,4s} \ne 0 \,|\, R[t_{2j}, t_{2j+1}]\forall j) \quad . \qquad (6)$$

Let $\gamma = \gamma_i$ be the (chronologically) first G elements of $\mathrm{LE}^+(R[-\infty, t_i]) \cap B(v, r)$ where G is some number. If $\mathrm{LE}^+(R[-\infty, t_i]) \cap B(v, r)$ contains less than G elements, take $\gamma = \mathrm{LE}^+(R[-\infty, t_i]) \cap B(v, r)$. (4) and (6) allow us to get a "second-order estimate for γ". By this we mean the quantity

$$V_s := \#\{w_1, w_2 \in \gamma : |w_1 - w_2| \le s\}$$

which has the estimate

$$\mathbb{P}(\#\gamma > \delta\mathbb{E}L_{i,v,r} \text{ and } V_s > \lambda \log(s/\delta)f(s)\#\gamma) \le Ce^{-c\lambda} \qquad (7)$$

for any parameters $\lambda > 0$ and $0 < \delta < 1$.

Before starting the proof of (7) let us just remark that the first condition and the variable δ are unfortunate technicalities. The "essentials" of (7) are really the

stronger claim $\mathbb{P}(V_s > \lambda(\log s)f(s)\#\gamma) \le Ce^{-c\lambda}$, but we don't know how to prove it. Also note that it is rather easy to show $\mathbb{P}(V_s > \lambda(\log r)f(s)\#\gamma) \le Ce^{-c\lambda}$, saving us all the mucking with δ later on, but this inequality will cost us a $\sqrt{\log r}$ in the final result of theorem 1.

Proof of (7). Cover $B(v, r)$ by balls $\{B_j\}$ of radius $2s$ such that any two points of distance $\le s$ are inside at least one B_j, and such that each point is covered $\le C$ times. Examine one $B_j = B(w_j, 2s)$. We have (not writing the "$| R[t_{2j}, t_{2j+1}]\forall j$" for brevity)

$$\mathbb{E}L_{i,v,r} > c\sum_j \mathbb{P}(L_{i,w_j,8s} > 0) \overset{(6)}{\ge} ce^{c\lambda}\sum_j \mathbb{P}(L_{i,w_j,2s} > \lambda f(s)) \quad \forall\lambda.$$

Denote by X_μ the total volume of the balls B_j where $L_{i,w_j,2s} > \mu f(s)$ and get $\mathbb{E}X_\mu \le Ce^{-c\mu}s^d\mathbb{E}L_{i,v,r}$. This gives, using $\mathbb{P}(X_\mu > e^{c\mu}\mathbb{E}X_\lambda) \le e^{-c\mu}$,

$$\mathbb{P}(X_\mu > Cs^d e^{-c\mu}\mathbb{E}L_{i,v,r}) \le e^{-c\mu} \quad \forall\mu$$

and shoving in $\#\gamma$ in a way that might look, for now, a little artificial, we get

$$\mathbb{P}(\#\gamma > \delta\mathbb{E}L_{i,v,r} \text{ and } X_\mu > Cs^d\delta^{-1}e^{-c\mu}\#\gamma) \le e^{-c\mu} \quad \forall\mu.$$

Taking $\mu_k = \lambda\log(s/\delta) + Ck$ and assuming that $\lambda > C$ for some C sufficiently large (as we may, without loss of generality), we get

$$\mathbb{P}(\#\gamma > \delta\mathbb{E}L_{i,v,r} \text{ and for some } k, \ X_{\mu_k} > c^{-k}\#\gamma) \le Ce^{-c\lambda} \quad . \tag{8}$$

Now, since $V_s \le \sum_j \#(\gamma \cap B_j) \cdot L_{i,w_j,2s}$, then

$$V_s \le \#\gamma(\lambda\log(s/\delta) + C)f(s) + \sum_{k=1}^\infty X_{\mu_k}(\lambda\log(s/\delta) + Ck)f(s).$$

If it happens that $X_{\mu_k} \le c^{-k}\#\gamma$ for all k i.e. the opposite of the second half of the event in (8), then

$$V_s \le \lambda\log(s/\delta)f(s)\#\gamma + \sum_{k=1}^\infty (c^{-k}\#\gamma)f(s)(\lambda\log(s/\delta) + Ck)$$
$$\le C\lambda\log(s/\delta)f(s)\#\gamma$$

and we get (7). This argument works for any $s \le \frac{1}{16}r$ but (7) holds for larger s too (there's not much point in $s > 2r$ of course) — we only have to pay in the constant C. $\qquad\square$

We want (7) to hold not for one particular s but for all s and the simplest version of such an inequality is

$$\mathbb{P}(\#\gamma > \delta\mathbb{E}L_{i,v,r} \text{ and } \exists s \text{ s.t. } V_s > \lambda\log^2(s/\delta)f(s)\#\gamma) \le Ce^{-c_1\lambda} \tag{9}$$

which follows from using (7) with $\lambda_s := \lambda\log(s/\delta)$ and summing over s.

Continuing the proof of the lemma, it is now time to examine the present. We keep the notations of G, γ and V_s. For an odd i we want to estimate the probability

$$p_i := \mathbb{P}(R[t_i, t_{i+1}] \cap \gamma \ne \emptyset) \quad .$$

Lemma A.5 allows us to consider a unconditioned random walk starting from $R(t_i)$ and stopped on $\partial B(v, 4r)$ instead of R. Denote it by R'. Denote by X_i the number of intersections of R' with γ, so $p_i \approx \mathbb{P}(X_i > 0)$. We have

$$\mathbb{E}(X_i \,|\, \text{past}) = \sum_{t=t_i}^{t_{i+1}} \sum_{w \in \gamma} \mathbb{P}(R'(t) = w \,|\, \text{past}) \quad .$$

For $r^2 \le t - t_i \le 2r^2$ we have for half of the $w \in B(v, r)$ that $\mathbb{P}(\{R'(t) = w\} \cap \{t < t_{i+1}\}) > cr^{-d}$ ("half of the w's" means that we need $t - t_i + ||w - R(t_i)||_1$ to be even, otherwise the probability is zero). Therefore

$$\mathbb{E}(X_i \,|\, \text{past}) > cr^{2-d}\#\gamma \quad . \tag{10}$$

Next estimate $\mathbb{E}(X_i^2 \,|\, \text{past})$. Assume until further notice that $V_s \le \lambda \log^2(s/\delta) \cdot f(s)\#\gamma$ for some δ and λ and for all s. Then

$$\mathbb{E}(X_i^2 \,|\, \text{past}) = \sum_{t_1, t_2, w_1, w_2} \mathbb{P}(R'(t_i) = w_i) \le \tag{11}$$

$$\le 2 \sum_{\Delta=0}^{\infty} \sum_{k=0}^{\infty} \sum_{t, w_1, w_2} \mathbb{P}\Big(R(t) = w_1, R(t + \Delta) = w_2,$$

$$k\sqrt{\Delta} \le |w_1 - w_2| < (k+1)\sqrt{\Delta} \Big)$$

Examine one couple of $w_1, w_2 \in \gamma$ with $k\sqrt{\Delta} \le |w_1 - w_2|$. Remembering the independence of the past from the present we can estimate the probability of one summand with a standard estimate on the end point of a random walk of length Δ starting from w_1. We get

$$\mathbb{P}(R(t) = w_1, R(t + \Delta) = w_2) \le Cr^{-d}\Delta^{-d/2}e^{-k^2/2} \quad .$$

We sum over all t. Since, easily, $\mathbb{P}(t_{i+1} - t_i > \Delta) \le Ce^{-c\Delta/r^2}$ and since $\mathbb{E}(t_{i+1} - t_i \,|\, t_{i+1} - t_i > \Delta) \le C \max\{r^2, \Delta\}$ we get

$$\sum_t \mathbb{P}(R(t) = w_1, R(t + \Delta) = w_2) \le Ce^{-c\Delta/r^2} \max\{r^2, \Delta\}r^{-d}\Delta^{-d/2}e^{-k^2/2} \quad .$$

Plugging this into (11) we get

$$\mathbb{E}(X_i^2 \,|\, \text{past}) \le C \sum_{\Delta=0}^{\infty} \sum_{k=0}^{\infty} e^{-c\Delta/r^2} \max\{r^2, \Delta\}r^{-d}\Delta^{-d/2}e^{-k^2/2}V_{(k+1)\sqrt{\Delta}} \tag{12}$$

For all our functions f (that is, all the specific functions we named in the statement of the lemma) we have

$$\sum_{k=0}^{\infty} e^{-k^2/2}V_{(k+1)\sqrt{\Delta}} \le \lambda\#\gamma \sum_{k=1}^{\infty} e^{-(k-1)^2/2}f(k\sqrt{\Delta}) \log^2(k\sqrt{\Delta}/\delta) \le$$

$$\le C\lambda\#\gamma f(\sqrt{\Delta}) \log^2(\Delta/\delta) \quad .$$

Similarly, for all our functions f we have

$$\sum_{\Delta=0}^{\infty} e^{-c\Delta/r^2} \max\{r^2, \Delta\} \Delta^{-d/2} f(\sqrt{\Delta}) \log^2(\Delta/\delta) \leq \tag{13}$$

$$\leq Cr^2 \sum_{\Delta=0}^{r^2} \Delta^{-d/2} f(\sqrt{\Delta}) \log^2(\Delta/\delta) \quad .$$

(12) and (13) give

$$\mathbb{E}(X_i^2 \,|\, \text{past}) \leq C\lambda r^{2-d} \#\gamma \sum_{\Delta=0}^{r^2} \Delta^{-d/2} f(\sqrt{\Delta}) \log^2(\Delta/\delta)$$

and then with (10) and the standard inequality $\mathbb{P}(X > 0) \geq (\mathbb{E}X)^2/\mathbb{E}X^2$ we get

$$\mathbb{P}(X_i > 0 \,|\, \text{past}) > c \frac{r^{2-d} \#\gamma}{\lambda \sum \Delta^{-d/2} f(\sqrt{\Delta}) \log^2(\Delta/\delta)} \quad . \tag{14}$$

This inequality is the heart of the proof. We recall that we assumed $V_s \leq \lambda \cdot \log^2(s/\delta) f(s) \#\gamma$ to get it.

Fix $G = \mu g(r)$ where $\mu > 1$ is some variable which we will fix later and where g is as defined in the beginning of the lemma. Let $H = 2 \lfloor g(r) r^{-2} \rfloor$ where $\lfloor \cdot \rfloor$ is the integer value. Let $X_1 = X_1(\mu)$ be the event that $\#\gamma_i = G$, let $X_2 = X_2(\lambda, \delta, \mu)$ be the event that $V_s \leq \lambda \log^2(s/\delta) f(s) G$ for all s (λ and δ are two additional variables) and let $X_3 = X_3(\mu)$ be the event that $R[t_{j+1} - t_j] \cap \gamma = \emptyset$ for all odd $i \leq j \leq i + H$. The events comprising X_3 are (conditioning on the $R(t_j)$) independent, therefore we may use (14) $\frac{1}{2}H$ times to get

$$\mathbb{P}(X_3 \,|\, X_1 \cap X_2) \leq \left(1 - c \frac{r^{2-d} \mu g(r)}{\lambda \sum_{\Delta=1}^{r^2} \Delta^{-d/2} f(\sqrt{\Delta}) \log^2(\Delta/\delta)}\right)^{\frac{1}{2}H} \leq 1 - \frac{c\mu}{\lambda \log^2 \delta^{-1}} \quad . \tag{15}$$

To see the rightmost inequality in (15), for each of the cases in the formulation of the lemma, apply the corresponding f and g and estimate the sum. Indeed, (15) is the inequality that governs the connection between f and g. Note that the formulation of the lemma is a little lax: if $f(r) = r^\alpha \log^\beta r$ with $\alpha > d - 2$ then we can actually prove the lemma with $g = r^{\alpha/2} \log^{(\beta+2)/2}$ i.e. one $\sqrt{\log r}$ factor better than the formulation of the lemma. This additional $\sqrt{\log r}$ factor is here only for the case $\alpha = d - 2$ and $\beta < -3$. Have no fear — this factor will disappear in the conclusion of theorem 1.

The proof of the lemma will now follow by induction over i. We use a "jumping induction" that assumes that for some k and K we have the inequality $\mathbb{P}(L_{i,v,r} > \nu g(r)) \leq Ke^{-k\nu}$ for all $\nu > 0$ and then proves the same for $L_{i+H,v,r}$ (the case $i = 0$ needs no explanation). Therefore we need first to calculate how much $L_{i,v,r}$ can change in between. Clearly, if $R([t_j, t_{j+1}])$ does not intersect $\mathrm{LE}([R[0, t_j])$ then

$$L_{j+1,v,r} - L_{j,v,r} \leq t_{j+1} - t_j \quad .$$

These variables have the simple estimate

$$\mathbb{P}(t_{j+1} - t_j > \nu r^2) \leq Ce^{-c\nu} \tag{16}$$

irrespectively of $R(t_{j+1})$ and $R(t_j)$ for all j odd. Denote by A_i the sum of $\frac{1}{2}H$ of those, and get a similar estimate (see lemma A.9):

$$\mathbb{P}\left(A_i > \nu g(r)\right) \leq Ce^{-c_2\nu} \quad A_i := \sum_{\substack{j=i \\ j\ \mathrm{odd}}}^{i+H} t_{j+1} - t_j \quad . \tag{17}$$

Next we make the following important assumption:

$$G > \delta\mathbb{E}L_{i,v,r} \quad \forall i. \tag{18}$$

Actually, we want it to be true independently of the value of μ, so we really need $g(r) > \delta\mathbb{E}L_{i,v,r}$. This holds for δ sufficiently small, but it is inconvenient to fix the value of δ at this point, as it depends on some constants (depending on d only) which are determined only later. Therefore we shall perform the necessary calculations with δ a variable and finally fix its value as some constant when we have all the information at hand, see (20). With a value of δ satisfying (20), or smaller, (18) will hold.

It is time to compare $L_{i,v,r}$ with $L_{i+H,v,r}$. $L_{i+H,v,r}$ might be larger than $\nu g(r)$ for the simple reason that A_i is very large. Let $\tau \leq \nu$ be yet another variable describing what "very large" means and we may estimate this phenomenon simply by

$$\sum_{n=\tau}^{\nu} \mathbb{P}(\{L_{i,v,r} > (\nu - n - 1)g(r)\} \cap \{A_i > ng(r)\}) \quad .$$

"Simply" because we ignore any effect of intersections. If, however, A_i is not as large we need both $L_{i,v,r}$ to be rather large, and X_3, i.e. to have no intersections with a path of length $G = \mu g(r)$ during the last H "moves". We need to assume $\mu + \tau < \nu$ for this to make sense, and this assumption holds until (19) below and we will not repeat it. All in all we get

$$\mathbb{P}(L_{i+H,v,r} > \nu g(r)) \leq \mathbb{P}(\{L_{i,v,r} > (\nu - \tau)g(r)\} \cap X_3) +$$

$$+ \sum_{n=\tau}^{\nu} \mathbb{P}(\{L_{i,v,r} > (\nu - n - 1)g(r)\} \cap \{A_i > ng(r)\}) \quad \forall i, \nu, \tau, \mu$$

(the parameter μ hides in the definition of X_3). For the first summand we have by (9), (18), (15) and the induction hypothesis that

$$\mathbb{P}(\{L_{i,v,r} > (\nu - \tau)g(r)\} \cap X_3) \leq$$

$$\leq \mathbb{P}(\{L_{i,v,r} > (\nu - \tau)g(r)\} \setminus X_2) + \mathbb{P}(\{L_{i,v,r} > (\nu - \tau)g(r)\} \cap X_3 \cap X_2)$$

$$\leq Ce^{-c_1\lambda} + Ke^{-k(\nu-\tau)}\left(1 - \frac{c\mu}{\lambda \log^2 \delta^{-1}}\right) \quad \forall i, \nu, \tau, \lambda, \mu, \delta$$

and estimating the other summands using (17) we get

$$\mathbb{P}(L_{i+H,v,r} > \nu g(r)) \leq Ke^{-k(\nu-\tau)}\left(1 - \frac{c\mu}{\lambda \log^2 \delta^{-1}}\right) + Ce^{-c_1\lambda} +$$

$$+ \sum_{n=\tau}^{\nu} Ke^{-k(\nu-n-1)} \cdot Ce^{-c_2 n} \quad \forall i, \nu, \tau, \lambda, \mu, \delta. \tag{19}$$

Having arrived at this closed formula, we only need to pick our variables carefully. First pick $\tau = \left\lfloor C\log\delta^{-1} \right\rfloor$ for some C sufficiently large. This will give, if $k < c_2/2$,

that

$$\sum_{n=\tau}^{\nu} Ke^{-k(\nu-n-1)} \cdot Ce^{-c_2 n} \le C\frac{e^{-C\log \delta^{-1}}}{1-e^{-c_2/2}}Ke^{-k(\nu-\tau)} \le C\delta Ke^{-k(\nu-\tau)} \quad .$$

Next we pick $\lambda = C\nu$ and $\mu = \frac{1}{2}\nu$, and the requirement $\mu + \tau < \nu$ translates to $\nu > C\log \delta^{-1}$. We get from everything that

$$\mathbb{P}(L_{i+H,v,r} > \nu g(r)) \le Ke^{-kv}\left(e^{kC\log \delta^{-1}}\left(1 - \frac{c}{\log^2 \delta^{-1}} + C\delta\right) + Ce^{-c\nu}\right)$$

Pick $k = c\log^{-3}\delta^{-1}$ and get, for δ sufficiently small and $\nu > C\log \delta^{-1}$ that

$$\mathbb{P}(L_{i+H,v,r} > \nu g(r)) \le Ke^{-k\nu}\left(1 - \frac{c}{\log^2 \delta^{-1}}\right) \quad .$$

Pick K sufficiently large so that the inequality $\mathbb{P}(L_{i,v,r} > \nu g(r)) \le Ke^{-k\nu}$ will hold trivially for $\nu \le C\log \delta^{-1}$ — notice that because $k = c\log^{-3}\delta^{-1}$ we have that K does not depend on δ — and our induction is complete. With these k and K, the inequality $\mathbb{P}(L_{i,v,r} > \nu g(r)) \le Ke^{-k\nu}$ is preserved from i to $i + H$ and since it clearly holds for $i \le H$ then it holds for all i.

Is this the end of the lemma? Almost. We still need to justify the assumption (18). The estimate $\mathbb{P}(L_{i,v,r} > \nu g(r)) \le Ke^{-k\nu}$ gives $\mathbb{E}L_{i,v,r} \le g(r)\frac{K}{k} \le Cg(r)\log^3 \delta^{-1}$. Therefore (remember that $G > g(r)$) the assumption reduces to the inequality

$$g(r) > g(r) \cdot (C\delta \log^3 \delta^{-1}) \quad . \tag{20}$$

Taking δ sufficiently small this will hold, and the lemma is proved. $\qquad\square$

Lemma 2. *The d-dimensional random walk has the f-property for*

$$f_d(r) := \begin{cases} r^{d/2} & d > 4 \\ r^{2+\epsilon} & d = 4 \end{cases} \quad . \tag{21}$$

Proof. Trivially, the d-dimensional random walk has the r^d-property. Therefore we may apply lemma 1 twice for $d > 6$, thrice for $d = 6$ or 5 and $\log \epsilon^{-1}$ times for $d = 4$. $\qquad\square$

Proof of theorem 1. Lemma 2 gives

$$\mathbb{P}(L_{i,v,r} > \lambda f(r)) \le Ce^{-c\lambda}$$

where $L_{i,v,r} = \#(\text{LE}(R[0, t_i]) \cap B(v,r))$ for any v and r satisfying $b \notin B(v, 2r)$, where f is defined by (21). Note that at this point we do not need the formulation in terms of continued process, and we may set the negative part of R to empty. If in addition $e \notin B(v, 4r)$ then the event that R is stopped between t_I and t_{I+1} is external to the ball, therefore we get that (21) holds for I. Since the section of the walk from t_I until the time when R hits e can only decrease $\text{LE}(R) \cap B(v,r)$ we get

$$\mathbb{P}(\#(L \cap B(v,r)) > \lambda f(r)) \le Ce^{-c\lambda}$$

However, we can cover our torus by balls $B(v_{i,j}, N2^{-i})$ with the property $b, e \notin B(v_{i,j}, 4N2^{-i})$ and with the number of j's corresponding to each i bounded by a

constant. Therefore for some constant c_3 sufficiently small we have

$$\mathbb{P}(\#L > \lambda f(N)) \leq \mathbb{P}(\exists i, j \text{ s.t. } L \cap B(v_{i,j}, r) > c_3 \lambda 2^{i/4} f(r)) \qquad (22)$$

$$\leq \sum_{i=0}^{c \log N} Ce^{-c\lambda 2^{i/4}} \leq Ce^{-c\lambda} \quad . \qquad \square$$

Remark. The same techniques can be improved to show that

$$\mathbb{P}(\#L > \lambda f(N)) \leq Ce^{-c\lambda^2}$$

where f is given by (21). The basic phenomenon behind this estimate is that to get a path of length $\lambda f(N)$, we need to have that each of the λ sections of the random walk, which are essentially independent, would not intersect any other. Since there are $c\lambda^2$ couples, the true estimate of the probability is square-exponential, as above. The analysis required to get this estimate is not inherently more difficult than that of the exponential estimate, but is more technical and we decided to represent the simpler exponential estimate.

On the other hand, we are not aware of a simpler version of the proof that gives an estimate of the decay of the probability worse than exponential. This follows from the recursive character of the proof. Thus, lemma 1 may be simplified by removing the requirement that the probability decays exponentially, but it then cannot be used recursively to get a reasonable final result. Similarly, the very strong independence condition in lemma 1, that the probability estimate inside every ball is independent of everything that happens outside the ball, cannot be relaxed without destroying the ability of the lemma to be used recursively.

We wish to reiterate that the only major simplification we are aware of of this proof is the one discussed after (7) (page 6). It saves the discussion after (5), i.e. the one leading to (6), as well as each and every appearance of the parameter δ. The cost is an added $\sqrt{\log}$ factor in the formulation of the theorem.

Conjecture. *The accurate upper bound in dimension* 4 *is*

$$N^2 \log^{1/6} N \quad .$$

The method above may be refined in many points and an estimate of the type $N^2 \log^\alpha N$ may be achieved for rather small α's. However, a fundamental difficulty is the fact that the sum in the denominator of (14) truly depends on N, which means that the second moment methods used here alone cannot give a precise result.

3. ABSOLUTE TIMES

The proof of the lower bound is, as will be seen in section 4, quite simple once a good estimate of the upper bound is available. Actually, one might think about the recursive nature of the proof of the upper bound in the following terms: "the proof of the upper bound was only possible once a good estimate of the upper bound was available".

Unfortunately, we were not able to get a reasonable proof of the lower bound using only lemma 1. The problem is that we need to know what happens at absolute times, i.e. to fix some t and get an estimate for $\text{LE}(R[0, t])$. Calculations true for t_i do not hold automatically for a fixed t. Apriori, one cannot rule out behavior such as "the loop-erased random walk is much denser if t is divisible by 1024",

since the t_i's might avoid those "bad absolute times". The purpose of this section is to show that this ridiculous behavior does not occur.

The first step is to learn something about the distribution of the t_i's. Since t_i is a sum of the return times to some sphere, and these return times are more-or-less independent, we would expect a central limit theorem. We don't need something so precise — we shall prove below (lemma 4) a large deviation estimate of the sort one would expect from a Gaussian variable, and this will be enough. We start with

Lemma 3. *Let* $X_1, \ldots, X_n$ *be variables with the properties*

$$\mathbb{P}(|X_i| > \lambda \,|\, X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n) \leq Ce^{-c\lambda} \tag{23}$$

$$\mathbb{E}(X_{i_1} \cdots X_{i_k} \,|\, X_{i_{k+1}}, \ldots, X_{i_l}) \leq \prod_{j=1}^{k} C \exp(-c \min_{\substack{1 \leq m \leq l \\ m \neq j}} |i_j - i_m|) \tag{24}$$

where (24) needs to hold only for $i_1, \ldots, i_l$ *all different. Then for all* $\lambda < cn^{1/4}$

$$\mathbb{P}\left(\left|\sum X_i\right| > \lambda\sqrt{n}\right) \leq Ce^{-c\lambda^2} \quad.$$

We interpret the condition (24) in the case $k = l = 1$ as saying $\mathbb{E}X_i = 0$ for all i. In the case $k > 1$, we call (24) a "pseudo independence" relation, because, rather than claiming that $\mathbb{E}\prod X_i = 0$, as we would have for independent variables, we get that it is exponentially small in the distance, so that if the i_k's are relatively sparse, it will be extremely small. Actually, it is possible to replace $\exp(-ck)$ with any sequence a_k with $\sum a_k < C$.

The proof is a pretty standard exercise: a calculation (which can be done either directly or by comparing to the case of independent exponential variables) can show that for $k < c\sqrt{n}$,

$$\mathbb{E}\left(\sum X_i\right)^{2k} \leq (Ckn)^k \quad.$$

Taking $k = c\lambda^2$ and using Markov's inequality will give the lemma. We skip the gory details.

Lemma 4. *Let* $b \in T_N^d$ *and let* R *be a random walk on* T *starting from* b. *Let* $C < r < \frac{1}{8}N$, $v \in T_N^d$ *and let* t_i *be the stopping times defined by (2). Then there exists numbers* $E = E(r) \approx N^d r^{2-d}$ *and* $\sigma = \sigma(r) \approx E$ *such that*

$$\mathbb{P}(|t_n - nE| > \lambda\sigma\sqrt{n}) \leq Ce^{-c\lambda^2} \tag{25}$$

for all $n \in \mathbb{N}$ *and* $\lambda < cn^{1/4}$.

Proof. The point is of course to show that the variables $t_{i+1} - t_i$ are pseudo independent and apply lemma 3. The first thing to note is that the distributions of $R(t_i)$ converge exponentially. Let q_1 and q_2 be two distributions on $\partial B(v, 2r)$, and denote

$$\epsilon := \sum_{x \in B(v, 2r)} |q_1(x) - q_2(x)| \quad.$$

Let R_μ, $\mu = 1, 2$ be random walks starting from a point on $\partial B(v, 2r)$ chosen with the distribution q_μ and stopped when hitting $\partial B(v, 4r)$. Let p_μ be the distributions on the hit points of R_μ. Then

$$p_1(w) - p_2(w) = \sum_{x \in \partial B(v, 2r)} (q_1(x) - q_2(x))\pi(x, w) \tag{26}$$

where $\pi(x, w)$ is the probability of a random walk starting from x to hit w. Let $A^+ \subset \partial B(v, 2r)$ be the set where $q_1(x) \geq q_2(x)$, and define

$$D^+(w) = \sum_{x \in A^+} |q_1(x) - q_2(x)| \pi(x, w) \quad .$$

Clearly

$$\sum_{w \in \partial B(v, 4r)} D^+(w) = \sum_{x \in A^+} \sum_w |q_1(x) - q_2(x)| \pi(x, w) = \frac{1}{2}\epsilon$$

and similarly for D^- defined equivalently using $A^- := \partial B(v, 2r) \setminus A^+$. Furthermore, the inequality $\pi(x, w) \approx r^{1-d}$ (see lemma A.4) gives that $D^{\pm}(w) \approx \epsilon r^{1-d}$ and therefore

$$|D^+(w) - D^-(w)| \leq (1 - c)(D^+(w) + D^-(w))$$

for some constant $c > 0$. This gives

$$\sum_{w \in \partial B(v, 4r)} |p_1(w) - p_2(w)| = \sum_w |D^+(w) - D^-(w)| \tag{27}$$

$$\leq (1 - c) \sum_w D^+(w) + D^-(w) = (1 - c)\epsilon$$

and we see that the L^1 distance between the distributions has contracted. An identical calculation works when the random walk starts from $\partial B(v, 4r)$ and stops at $\partial B(v, 2r)$ (see the remark following lemma A.4) therefore we see that there is only one limiting distribution as i increases, and that the L^1 distance to this distribution decreases exponentially with i. In other words, if t_i^{μ} are stopping times defined by (2) for the walks R_{μ} then we get

$$\sum_w |\mathbb{P}(R_1(t_i^1) = w) - \mathbb{P}(R_2(t_i^2) = w)| \leq \epsilon e^{-ci} \quad . \tag{28}$$

This L^1 estimate allows to get a uniform estimate for every w and $i > 0$:

$$|\mathbb{P}(R_1(t_i^1) = w) - \mathbb{P}(R_2(t_i^2) = w)| \leq Ce^{-ci} \min_{\mu=1,2} \mathbb{P}(R_{\mu}(t_i^{\mu}) = w). \tag{29}$$

Indeed, take the distributions of R_{i-1}^{μ} as the q_{μ}'s in (26) and together with (28) and $\pi(x, w) \leq Cr^{1-d}$ get that

$$|\mathbb{P}(R_1(t_i^1) = w) - \mathbb{P}(R_2(t_i^2) = w)| \leq Cr^{1-d}e^{-ci} \quad .$$

In the other direction, $\pi(x, w) \geq cr^{1-d}$ gives $\mathbb{P}(R_{\mu}(t_i^{\mu}) = w) \geq cr^{1-d}$ and we get (29).

To make notations simpler, let B_i be $\partial B(v, 2r)$ if i is odd and $\partial B(v, 4r)$ if i is even. Now, each $t_{i+1} - t_i$ has an exponential distribution[2], with its expectation being less or equal than

$$U_i := \begin{cases} Cr^2 & i \text{ is odd} \\ CN^d r^{2-d} & i \text{ is even} \end{cases} \tag{30}$$

---

[2]For i even, $t_{i+1} - t_i$ has a rather large ($> c$) probability to be very small, of the order of r^2. However, since there is also a probability $> c$ to escape $B(v, \frac{1}{2}N)$, this fact has negligible impact on the moments of $t_{i+1} - t_i$.

even after conditioning on the entry and exit points. In a formula,

$$\mathbb{P}(t_{i+1} - t_i > \lambda U_i \,|\, R(u_i) = y_1 \text{ and } R(u_{i+1}) = y_2) \le Ce^{-c\lambda} \tag{31}$$

for every $y_1 \in B_i$ and $y_2 \in B_{i+1}$ (see lemmas A.8 and A.11).

Define the variables

$$X_i := (t_{i+1} - t_i - \mathbb{E}(t_{i+1} - t_i))/U_0 \quad.$$

We wish to use lemma 3 for the X_i's. To get (23) we use (31) to see that $\mathbb{E}(t_{i+1} - t_i)/U_0 \le CU_i/U_0 \le C$ and then use (31) again to get

$$\mathbb{P}(X_i > \lambda \,|\, R(t_i), R(t_{i+1})) \le Ce^{-c\lambda} \quad. \tag{32}$$

Denote by $\mathcal{X}$ the event $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n$ and then

$$\mathbb{P}(X_i > \lambda \,|\, \mathcal{X}) = \mathbb{E}\mathbb{P}(X_i > \lambda \,|\, R(t_i), R(t_{i+1}), \mathcal{X}) = \mathbb{E}\mathbb{P}(X_i > \lambda \,|\, R(t_i), R(t_{i+1}))$$

$$\le \mathbb{E}Ce^{-c\lambda} = Ce^{-c\lambda}$$

where the expectation above is with respect to $R(t_i)$ and $R(t_{i+1})$. This gives (23).

The argument for (24) requires the convergence of the distributions. Start with the case of one i. Denote by $\mathcal{Y}$ the event $R(t_i), R(t_{i+1})$ and by $\mathcal{Z}$ the event $R(t_{i-\Delta})$, $R(t_{i+1+\Delta})$ for some $\Delta \in \{0, 1, \ldots\}$. Then

$$\mathbb{E}(X_i \,|\, \mathcal{Z}) = \sum_{y \in B_i \times B_{i+1}} \mathbb{P}(\mathcal{Y} = y \,|\, \mathcal{Z}) \cdot \mathbb{E}(X_i \,|\, \mathcal{Y} = y)$$

$$= \sum_{y \in B_i \times B_{i+1}} \big(\mathbb{P}(\mathcal{Y} = y \,|\, \mathcal{Z}) - \mathbb{P}(\mathcal{Y} = y)\big) \cdot \mathbb{E}(X_i \,|\, \mathcal{Y} = y) \tag{33}$$

$$\le C \sum_{y \in B_i \times B_{i+1}} \big|\mathbb{P}(\mathcal{Y} = y \,|\, \mathcal{Z}) - \mathbb{P}(\mathcal{Y} = y)\big| \tag{34}$$

where the equality (33) is due to $\mathbb{E}X_i = 0$. Denote by $\pi_k(w, x)$ the probability to start from w and hit x after k moves of going from B_j to B_{j+1}. In a formula

$$\pi_k(w, x) := \mathbb{P}(R(u_{j+k}) = x \,|\, R(u_j) = w) \quad.$$

Of course, we mean that if $w \in \partial B(v, 2r)$ then we take j odd and in the opposite case we take $j > 0$ even. Other than that the value of π_k is independent of j. With these notations we get

$$\mathbb{P}(\mathcal{Y} = (y_1, y_2)) = \mathbb{P}(R(t_i) = y_1)\pi_1(y_1, y_2)$$

$$\mathbb{P}(\mathcal{Y} = (y_1, y_2) \,|\, \mathcal{Z} = (z_1, z_2)) = \frac{\pi_\Delta(z_1, y_1)\pi_1(y_1, y_2)\pi_\Delta(y_2, z_2)}{\pi_{2\Delta+1}(z_1, z_2)}$$

so

$$|\mathbb{P}(\mathcal{Y} = y \,|\, \mathcal{Z} = z) - \mathbb{P}(\mathcal{Y} = y)| \le \pi_1(y_1, y_2)\bigg(|\mathbb{P}(R(t_i) = y_1) - \pi_\Delta(z_1, y_1)| + \tag{35}$$

$$+ \left|\frac{\pi_\Delta(y_2, z_2)}{\pi_{2\Delta+1}(z_1, z_2)} - 1\right|\pi_\Delta(z_1, y_1)\bigg) \quad.$$

Summing over y the first half of (35) we get

$$\sum_{y_1, y_2} \pi_1(y_1, y_2)|\mathbb{P}(R(t_i) = y_1) - \pi_\Delta(z_1, y_1)| = \tag{36}$$

$$= \sum_{y_1} |\mathbb{P}(R(t_i) = y_1) - \pi_\Delta(z_1, y_1)| \le 2e^{-c\Delta}$$

where the last inequality is due to the exponential convergence of the distributions in the form (28) — take q_1 to be the distribution of $R(t_{i-\Delta})$ and $q_2 = \delta_{\{z_1\}}$ (the distance between any two distributions is always ≤ 2). For the second half of (35), we use the form (29) for and get, under the assumption $\Delta > 0$,

$$\sum_{y_1, y_2} \pi_\Delta(z_1, y_1) \pi_1(y_1, y_2) \left| \frac{\pi_\Delta(y_2, z_2)}{\pi_{2\Delta+1}(z_1, z_2)} - 1 \right| \leq$$

$$\leq Ce^{-c\Delta} \sum_{y_1, y_2} \pi_\Delta(z_1, y_1) \pi_1(y_1, y_2) = Ce^{-c\Delta} \quad . \tag{37}$$

We used here (29) with $q_1 = \delta_{\{y_2\}}$ and q_2 the distribution of $R(u_{j+\Delta+1}) \,|\, R(u_j) = z_1$ for the point z_2. Using (36), (37) and (35) in (34) gives

$$\mathbb{E}(X_i \,|\, R(t_{i-\Delta}), R(t_{i+1+\Delta})) \leq Ce^{-c\Delta} \quad . \tag{38}$$

(the case $\Delta = 0$ doesn't follow from the argumentation above, but can be deduced, say, from (32)).

With (38), proving (24) is easy. Let $i_1, \ldots, i_l$ be some integers, all different, and let

$$\Delta_j = \left\lfloor \frac{1}{2} \left(\min_{\substack{1 \leq m \leq l \\ m \neq j}} |i_j - i_m| - 1 \right) \right\rfloor \quad ,$$

so that the intervals $]i_j - \Delta_j, i_j + 1 + \Delta_j[$ are disjoint. Let $\mathcal{X}$ be the event

$$R(t_{i_1-\Delta_1}), R(t_{i_1+1+\Delta_1}), \ldots, R(t_{i_k-\Delta_k}), R(t_{i_k+1+\Delta_k}) \quad .$$

Then conditioning by $\mathcal{X}$ the events X_i are independent so we get

$$\mathbb{E}(X_{i_1} \cdots X_{i_k} \,|\, \mathcal{X}) = \prod_{j=1}^{k} \mathbb{E}(X_j \,|\, \mathcal{X}) = \prod_{j=1}^{k} \mathbb{E}(X_j \,|\, R(t_{i_j-\Delta_j}), R(t_{i_j+1+\Delta_j}))$$

$$\overset{(38)}{\leq} \prod_{j=1}^{k} Ce^{-c\Delta_j} \leq \prod_{j=1}^{k} C \exp\left(-c \min_{\substack{1 \leq m \leq l \\ m \neq j}} |i_j - i_m|\right)$$

which immediately gives (24) since

$$\mathbb{E}(X_{i_1} \cdots X_{i_k} \,|\, X_{i_{k+1}}, \ldots, X_{i_l}) =$$

$$= \mathbb{E}\big(\mathbb{E}(X_{i_1} \cdots X_{i_k} \,|\, \mathcal{X}) \,\big|\, X_{i_{k+1}}, \ldots, X_{i_l}\big) \leq$$

$$\leq \mathbb{E}\left(\prod_{j=1}^{k} C \exp\left(-c \min_{\substack{1 \leq m \leq l \\ m \neq j}} |i_j - i_m|\right) \,\bigg|\, X_{i_{k+1}}, \ldots, X_{i_l} \right) =$$

$$= \prod_{j=1}^{k} C \exp\left(-c \min_{\substack{1 \leq m \leq l \\ m \neq j}} |i_j - i_m|\right)$$

with (23) and (24) established we can invoke lemma 3 and get

$$\mathbb{P}(|t_n - \mathbb{E}t_n| > \lambda U_0) \leq Ce^{-c\lambda^2} \quad .$$

Lemma 4 now follows since (29) shows that $\mathbb{E}t_{2i+1} - t_{2i}$ converge exponentially to some E_{even} and $\mathbb{E}t_{2i+2} - t_{2i+1}$ converge exponentially to some E_{odd} so

$$\left| \mathbb{E}t_n - n\tfrac{1}{2}(E_{\text{even}} + E_{\text{odd}}) \right| \leq CU_0$$

and for $\lambda < C\sqrt{n}$ this translation affects only the multiplicative constant. Therefore taking $E = \frac{1}{2}(E_{\text{even}} + E_{\text{odd}}) \approx N^d r^{2-d}$ and $\sigma = U_0 \approx E$ we are done. $\qquad \square$

Lemma 5. *Let $b \in T_N^d$ and let R be a random walk on T starting from b. Let $r < \frac{1}{8}N$, $v \in T_N^d$. Let $t \in \mathbb{N}$ be some time. Then*

$$\mathbb{P}(\text{LE}(R[0,t]) \cap B(v,r) > \lambda f(r)) \le Ce^{-c\lambda} \quad \forall \lambda > 0 \tag{39}$$

where f is defined by (21).

Proof. Let $\lambda > 0$ be some number. We note that we may assume $t < \lambda N^{d/2}$ since in time $\lambda N^{d/2}$ the probability to hit b is $> 1 - Ce^{-c\lambda}$ and in this case the process starts afresh, memoryless. Let t_i be stopping times defined by (2). Let E and σ be defined by lemma 4 so that (25) holds.

The first case is $\lambda > C_1 \log r$ for some C_1 sufficiently large. This case is uninteresting for the following reason: lemma 4 gives that for some C_2 sufficiently large, if $n = \lfloor C_2 \lambda r^{d-2} \rfloor$ then $\mathbb{P}(t_n \le t) \le Ce^{-c\lambda}$. Let $k := \max\{l : t_l \le t\}$. If k is even then $\text{LE}(R[0,t]) \cap B(v,r) \subset \text{LE}(R[0,t_k]) \cap B(v,r)$. If k is odd then

$$\#((\text{LE}(R[0,t]) \setminus \text{LE}(R[0,t_k])) \cap B(v,r)) \le t_{k+1} - t_k$$

and this variable has the estimate (16) so it is uninteresting. Therefore it is enough to calculate the loop-erased at the times t_k. We get

$$\mathbb{P}(\#(\text{LE}(R[0,t]) \cap B(v,r)) > \lambda f(r)) \le$$
$$\le Ce^{-c\lambda} + \sum_{k=1}^{n} \mathbb{P}(\#(\text{LE}(R[0,t_k]) \cap B(v,r)) > \tfrac{1}{2}\lambda f(r)) \le$$
$$\le Cne^{-c\lambda} \le Cr^{d-2}\lambda e^{-c\lambda} \le Cr^{d-2}e^{-c\lambda} \le Cr^{d-2-cC_1}e^{-c\lambda}.$$

(of course, all c's in the last line are different). This shows that for C_1 sufficiently large — namely, $(d-2)/c$ where c is the last c on the last line above, (39) holds. Thus this case is proved.

Therefore we shall assume that $\lambda < C \log r$. Let n_1 be defined by

$$n_1^- := \max\{n \text{ even} : t - nE > \lambda\sigma\sqrt{n}\}$$
$$n_1^+ := \min\{n : t - nE < -\lambda\sigma\sqrt{n}\}$$

Note that

$$n_1^+ - n_1^- \le C\lambda\sqrt{t/E} \le CN^{d/2}\lambda^{3/2}\sqrt{r^{d-2}}.$$

Let $\mathcal{E}_1$ be the event $|t_{n_1^-} - n_1^- E| > \lambda\sigma\sqrt{n_1^-}$ or $|t_{n_1^+} - n_1^+ E| > \lambda\sigma\sqrt{n_1^+}$. Lemma 4 gives us that $\mathbb{P}(\mathcal{E}_1) < Ce^{-c\lambda}$. We note that under $\neg\mathcal{E}_1$ we can "locate" t, $t_{n_1^-} < t < t_{n_1^+}$ and the interval is not very large, $t_{n_1^+} - t_{n_1^-} < CN^d\lambda^{3/2}\sqrt{r^{2-d}}$. Let $\mathcal{E}_2$ be the event $\#\text{LE}(R[0,t_{n_1}]) \cap B(v,r)) > \lambda f(r)$. Lemma 2 gives us that $\mathbb{P}(\mathcal{E}_2) < Ce^{-c\lambda}$.

We continue to define a short sequence of $n_j^\pm$ inductively:

$$n_j^- := \max\{n \text{ even} : t - t_{N_{j-1}^-} - nE > \lambda\sigma\sqrt{n}\}$$
$$n_j^+ := \min\{n : t - t_{N_{j-1}^-} - nE < -\lambda\sigma\sqrt{n}\}$$
$$N_j^\pm := n_j^\pm + \sum_{k=1}^{j-1} n_k^-$$

Unlike $n_1^\pm$ which are just numbers, $n_j^\pm$, $j > 1$ are events depending on $R[0, t_{N_{i-1}^-}]$.
Define $\mathcal{E}_{2i-1}$ to be the event $|t_{N_i^\pm} - t| > \lambda\sigma\sqrt{n_i^\pm}$ (as before, we mean that either
happens). Again, we get $\mathbb{P}(\mathcal{E}_{2i-1}) < Ce^{-c\lambda}$. Under $\neg(\mathcal{E}_1 \cup \mathcal{E}_3 \cup \cdots \cup \mathcal{E}_{2i-3})$ we have

$$n_i^\pm < C\frac{t - t_{N_{i-1}^-}}{E} \le C\frac{t_{N_{i-1}^+} - t_{N_{i-1}^-}}{E} \overset{(41)}{\le} C(i)\lambda^{2-2^{-i+1}} r^{2^{-i+1}(d-2)} \tag{40}$$

(the use of (41) is inductively, for $i-1$). The addition of $\neg\mathcal{E}_{2i-1}$ gives $t_{N_i^-} < t < t_{N_i^+}$
and

$$t_{N_i^+} - t_{N_i^-} \le \lambda\sigma\left(\sqrt{n_i^+} + \sqrt{n_i^-}\right) \overset{(40)}{\le} C(i)N^d\lambda^{2-2^{-i}} r^{(1-2^{-i})(2-d)} \quad . \tag{41}$$

To use lemma 2, we need to define an auxiliary walk R',

$$R_i'(u) = R(u + t_{N_{i-1}^-}) \quad R_i' : \{-t_{N_{i-1}^-}, \ldots, t - t_{N_{i-1}^-}\} \to T.$$

In other words, we consider the part of the walk until $t_{N_{i-1}^-}$ as fixed, and the part
from $t_{N_{i-1}^-}$ to t as the probabilistic part. Of course, the stopping times t_j' corre-
sponding to R_i' are simply $t_j' = t_{N_{i-1}^-+j}$. The fact that N_i^- is even means that
$R_i'(0) \notin B(v, 2r)$ and then lemma 2 will give that

$$\mathbb{P}(\mathcal{E}_{2i}) \quad \le \quad Ce^{-c\lambda}$$
$$\mathcal{E}_{2i} \quad := \quad \{\#(L_i \cap B(v, r)) > \lambda f(r)\} \tag{42}$$
$$L_i \quad := \quad \mathrm{LE}^+(R_i'[-t_{N_{i-1}^-}, t_{N_i^-} - t_{N_{i-1}^-}]) \tag{43}$$

Note that we have now defined all the exceptional events $\mathcal{E}_i$: the even ones are (42)
and the odd ones have been defined slightly above.

When we said that the series $n_i^\pm$ is short, we meant that we shall take it until I
defined by

$$I = \begin{cases} 2 & d \ge 7 \\ 3 & d = 5, 6 \\ C\log\epsilon^{-1} & d = 4 \end{cases}$$

where ϵ is from (21), which we consider as a constant, so $I \le C$. In particular

$$\mathbb{P}(\mathcal{E}_1 \cup \cdots \cup \mathcal{E}_{2i}) \le CIe^{-c\lambda} \le Ce^{-c\lambda}.$$

The reason for this selection of I is that with this I it is possible to do a simple
estimate of the path between $t_{N_I^-}$ and t. For any i we have

$$N_i^+ - N_i^- < C\lambda\sqrt{(t - t_{N_{i-1}^-})/E} \le C\lambda\sqrt{(t_{N_{i-1}^+} - t_{N_{i-1}^-})/E}$$

$$\overset{(41)}{\le} C\lambda^{2-2^{-i}} r^{2^{-i}(d-2)} \le Cr^{2^{-i}(d-2)}\log^2 r$$

(remember that $\lambda < C\log r$) and for I this gives

$$N_I^+ - N_I^- \le Cf(r)r^{-2} \quad .$$

Therefore we may use (16) $N_I^+ - N_I^-$ times, to get

$$\mathbb{P}\left(\sum_{\substack{j=N_I^- \\ j \text{ odd}}}^{N_I^+} t_{j+1} - t_j > \lambda f(r)\right) \le Ce^{-c\lambda} \tag{44}$$

which of course bounds also $\#(\mathrm{LE}(R[0,t]) \cap B(v,r)) - \#(\mathrm{LE}(R[0,t_{N_I^-}]) \cap B(v,r))$.
Finally, the definitions of R'_i, LE, LE^+ and L_i (43) give

$$\mathrm{LE}(R[0,t_{N_I^-}]) \subset L_1 \cup L_2 \cup \cdots \cup L_I$$

and assuming $\neg(\mathcal{E}_2 \cup \mathcal{E}_4 \cup \cdots \cup \mathcal{E}_{2i})$ we have from (42) that

$$\#(\mathrm{LE}(R[0,t_{N_I^-}]) \cap B(v,r)) \leq I\lambda f(r) \leq C\lambda f(r)$$

and with (44) we finally get

$$\mathbb{P}(\#(\mathrm{LE}(R[0,t]) \cap B(v,r)) > \lambda f(r)) \leq Ce^{-c\lambda} \quad .$$

and the lemma is proved. $\qquad \square$

Remark. By now the reader would not be surprised to learn that here too, if one is willing to let go of a $\log$ factor then the proof gets much simpler. Indeed, the arguments used for the case $\lambda > C \log r$ can be used for any λ to get this result, and for this case one does not need the precise estimates of lemma 4 either, and the entire section may be reduced to half a page.

Theorem 3. *Let $b \in T_N^d$ and let R be a random walk on T starting from b. Let $t \in \mathbb{N}$ be some time. Then*

$$\mathbb{P}(\mathrm{LE}(R[0,t]) > \lambda f(N)) \leq Ce^{-c\lambda} \quad \forall \lambda > 0$$

where f is defined by (21).

The theorem follows from lemma 5 like theorem 1 follows from lemma 2 (cover T by balls etc.) and we shall omit the proof.

4. THE LOWER BOUND

We will use the concept of a cut time

Definition. Let R be a random walk on a graph, possibly with a stopping condition. A time t is called a **cut time** for R if $R[0,t] \cap R]t,\infty[= \emptyset$.

Clearly, if t is a cut time then $R(t) \in \mathrm{LE}(R)$. Further, all $R(t_i)$'s for different cut times t_i are different. Therefore it is possible to estimate the length of a loop-erased random walk by counting cut times.

Lemma 6. *Let $d \geq 5$. Let R be a random walk on T_N^d of length L for some $L = \epsilon N^{d/2}$, ϵ sufficiently small and $N > N_0(\epsilon)$. Let X be the number of cut times of R. Then*

$$\mathbb{E}X > cL \quad \mathbb{V}X < C\epsilon^2 L^2 \quad . \tag{45}$$

As usual $\mathbb{V}$ denotes the variance, i.e. $\mathbb{V}X := \mathbb{E}X^2 - (\mathbb{E}X)^2$.

Proof. Denote by E_t the event that t is a cut time. Easily,

$$1 - \mathbb{P}(E_t) \leq \sum_{s_1=0}^{t} \sum_{s_2=t+1}^{L} \mathbb{P}(R(s_1) = R(s_2)) \quad .$$

Now for $|s_i - t| \leq N$ this is identical to the equivalent problem on $\mathbb{Z}^d$ which is well known (see [L96]) so we get

$$\mathbb{P}(R[\max 0, t-N, t] \cap R]t, \min L, t+N] \neq \emptyset) < 1 - c \quad .$$

For other s_i we use the easy

$$\mathbb{P}(R(s_1) = R(s_2)) \leq C \min\{N^2, |s_1 - s_2|\}^{-d/2} \tag{46}$$

to get

$$1 - \mathbb{P}(E_t) < 1 - c + C\epsilon^2 + CN^{2-d/2}$$

therefore for ϵ sufficiently small and N sufficiently large we get $\mathbb{P}(E_t) > c$ which gives the first part of (45) — $\mathbb{E}X > cL$. For the second part, we examine the covariance of E_{t_1} and E_{t_2} for some $t_1 < t_2$. Denote $t = \lfloor \frac{1}{2}(t_1 + t_2) \rfloor$ and

$$E_1' = \mathbb{P}(R[0, t_1] \cap \,]t_1, t] = \emptyset) \quad E_2' = \mathbb{P}(R[t, t_2] \cap R\,]t_2, L] = \emptyset) \quad .$$

We note that E_1' and E_2' are independent and therefore $\operatorname{cov} E_1', E_2' = 0$. On the other hand, summing (46) we get

$$|\mathbb{P}(E_1') - \mathbb{P}(E_{t_1})| \leq \mathbb{P}(R[0, t_1] \cap R[t, L] \neq \emptyset) \leq \sum_{s_1=0}^{t_1} \sum_{s_2=t}^{L} C \min\{N^2, |s_2 - s_1|\}^{-d/2}$$

$$\leq C \sum_{s_1=0}^{t_1} |t - s|^{1-d/2} + \epsilon N^{-d/2} \leq C(|t_2 - t_1|^{2-d/2} + \epsilon^2)$$

so we get the same for the covariance of E_{t_i},

$$\operatorname{cov} E_{t_1}, E_{t_2} \leq C|t_2 - t_1|^{2-d/2} + C\epsilon^2 \quad .$$

Summing these for all t_i's we get the second half of the lemma. $\qquad\square$

Lemma 7. *Let $d \geq 5$. Let $b \in T_N^d$ and let R be a random walk on T starting from b. Let $t \in \mathbb{N}$, $t > N^{d/2}$ and $\lambda > N^{-1/2}$. Then*

$$\mathbb{P}(\# \operatorname{LE}(R[0, t]) \leq \lambda N^{d/2}) \leq C\lambda$$

Proof. We may assume without loss of generality that $\lambda \leq c$ for some constant. Let C_1 be some constant which will be fixed later. Define

$$u := t - C_1 \lambda N^{d/2}.$$

(we assume here $\lambda < 1/C_1$, as we may). Denote by X the number of cut times in the segment $[u, t]$. Lemma 6 shows that $\mathbb{E}X > c(t - u) = cC_1 \lambda N^{d/2}$. Pick C_1 sufficiently large such that $\mathbb{E}X > 3\lambda N^{d/2}$. Lemma 6 also gives $\mathbb{V}X \leq C\lambda^4 N^{d/2}$ and then

$$\mathbb{P}(X \leq 2\lambda N^{d/2}) \leq C\lambda^2.$$

Next we want to estimate

$$\mathbb{P}(\operatorname{LE}(R[0, u]) \cap R[u + N^2, t] \neq \emptyset.)$$

Define $Y = \#\{\operatorname{LE}(R[0, u]) \cap R[u + N^2, t]\}$. If we assume $\# \operatorname{LE}(R[0, u]) \leq \mu N^{d/2}$, then because $R(u + N^2)$ is distributed $\approx$ uniformly on T we get

$$\mathbb{E}(Y \mid \# \operatorname{LE}(R[0, u]) \leq \mu N^{d/2}) \approx N^{-d}(\# \operatorname{LE}(R[0, u]))(t - u - N^2) \approx \mu\lambda$$

(this is the only place we use the assumption $\lambda > N^{-1/2}$). Without the assumption $\# \operatorname{LE}(R[0, u]) \leq \mu N^{d/2}$ we get

$$\mathbb{E}Y \leq \sum_{\mu=0}^{\infty} \mathbb{P}(\# \operatorname{LE}(R[0, u]) \leq \mu N^{d/2}) \cdot \mathbb{E}(Y \mid \# \operatorname{LE}(R[0, u]) \leq (\mu + 1)N^{d/2})$$

$$\leq \sum_{\mu=0}^{\infty} Ce^{-c\mu}\mu\lambda \leq C\lambda$$

and hence $\mathbb{P}(Y > 0) \leq C\lambda$. Under the assumption $Y = 0$ every cut point of $R[u, t]$ above $u + N^2$ is in $\mathrm{LE}(R[0, t])$ and the lemma follows.   □

Proof of theorem 2. Let R' be a random walk starting from b with no stopping condition. Define events

$$\mathcal{X}(v, t) = \{R'(t) = v \wedge v \notin R' [0, t[\}$$
$$\mathcal{Y}(v, t) = \{R'(t) = v \wedge \# \mathrm{LE}(R[0, t]) \leq \lambda N^{d/2}\}.$$

Now, $\sum_v \mathbb{P}(\mathcal{X}(v, t))$ is simply the probability that a random walk reaches its end point for the first time, or equivalently by symmetry, the probability that it never returned to its starting point, therefore it is easy to calculate

$$\sum_{v \in T} \mathbb{P}(\mathcal{X}(v, t)) \leq Ce^{-ctN^{-d}} \quad \forall t.$$

Next, for $t > N^{d/2}$, lemma 7 gives

$$\sum_{v \in T} \mathbb{P}(\mathcal{Y}(v, t)) \leq C\lambda \quad \forall t > N^{d/2}.$$

Finally, note that

$$\sum_{t=0}^{\infty} \mathbb{P}(\mathcal{X}(v, t)) = 1 \quad \forall v \in T.$$

With these three facts we get, for any parameter $\mu > 0$,

$$\sum_{t,v} \mathbb{P}(\mathcal{X}(v, t) \setminus \mathcal{Y}(v, t)) \geq$$

$$\geq N^d - \left(\sum_{t=0}^{N^{d/2}} + \sum_{t=\mu N^d}^{\infty} \right) \sum_{v \in T} \mathcal{P}(\mathcal{X}(v, t)) - \sum_{t=N^{d/2}}^{\mu N^d} \sum_{v \in T} \mathbb{P}(\mathcal{Y}(v, t)) \geq$$

$$\geq N^d(1 - Ce^{-c\mu} - N^{-d/2} - C\mu\lambda).$$

Picking $\mu = C \log \lambda$ for some C sufficiently large will prove the theorem.   □

4.1. **Remarks on alternative approaches.** The first alternative approach to the proof of the lower bound is as follows: prove a conditioned version of lemma 6, namely

Lemma. *Let b and e be two points on T_N^d with $|b - e| > cN$. Let R be a random walk on T_N^d of length L for some $L = \epsilon N^{d/2}$, ϵ sufficiently small starting from b and conditioned to end at e. Let X be the number of cut times of R. Then*

$$\mathbb{E}X > cL \quad \mathbb{V}X < C\epsilon^2 L^2 \quad .$$

This lemma allows to prove a version of theorem 2 for any points far enough, not just two random points. Further, it allows to avoid the need to use absolute times, and just work directly with the times t_i for some arbitrary ball. In other words, to show that the loop-erased random walk from b to e is long with high probability, define an arbitrary ball B, show that at the stopping times t_i corresponding to B the entire loop-erased random walk is quite small (this is quite simple) and then show that the random walk from the last t_i to e has many cut points using the lemma above.

The proof of this lemma requires no new ideas when compared with lemma 6. However, it is very technical, and quite long, which is the main reason we chose the approach above. In some sense we do not consider the length of section 3 as an indication that the approach we chose is more complicated because the result (theorem 3) is trivial if one can afford to lose a $\log$ factor (and also because the result is quite natural).

Another approach is the use of the uniform spanning tree and Wilson's algorithm (see [W96]). Roughly, one might hope to show that the loop-erased random walk is long by constructing an appropriate partial UST, and then showing that the random walk R starting from some point b and stopped on the partial UST is not too long (therefore no complicated self interactions, as in lemma 6) and not too short, so $\mathrm{LE}(R)$ can be proved to be long. Since the loop-erased random walk from b to some other point e (say inside the partial UST) contains $\mathrm{LE}(R)$, this will be enough. Alternatively, one can take two random walks R and R' starting from b and e respectively and stopped on the partially constructed UST, and calculate the probabilities that at least one is long and that they do not intersect. Both approaches allow to generalize theorem 2 from a random end point to any end point (naturally, if b and e are very close then with positive probability the loop-erased random walk from b to e is short. However, one can show that there is a positive probability for the loop-erased random walk to be long, i.e. $\approx N^{d/2}$).

A third strategy using the UST is as follows. Notice that the harmonic measure on a partially constructed UST is roughly uniform — this follows since the escape probabilities from a typical small ball are positive. If one wants to estimate the probability that the loop-erased random walk between b and e is $\leq \lambda N^{d/2}$, construct a partial UST containing b up until its size is $\approx (1/\lambda)N^{d/2}$, and then estimate that the number of vertices in the tree with distant $\leq \lambda N^{d/2}$ from b is $\approx \lambda N^{d/2}$, so the harmonic measure is $\approx \lambda^2$. This approach gives (in addition to the fact that e may be arbitrary) stronger estimates than the $\lambda \log \lambda^{-1}$ of theorem 2 — formalizing these arguments we were able to show $\mathbb{P}(\# \mathrm{LE}(R) \leq \lambda N^{d/2}) \leq C\lambda^2 \log \lambda^{-1}$, and we believe that the true value is, as in the case of the complete graph, λ^2. The difference between $\lambda \log \lambda^{-1}$ and $\lambda^2 \log \lambda^{-1}$ is significant in the following sense: the weaker estimate does not prove that the UST has a true branching nature: even points that are distributed linearly along a path of length $N^{d/2}$ satisfy the requirement that $\mathbb{P}(\mathrm{LE} \leq \lambda N^{d/2}) \leq C\lambda$. However, the estimate $\lambda^2 \log \lambda^{-1}$ allows to deduce non-trivial facts about the branching structure of the UST.

None of these methods work in dimension 4, and the culprit is always the same: in dimension 4 our methods do not show that within the mixing time the probability of hitting the loop-erased random walk is small. In other words, to get a lower bound for dimension 4 one must either show a very precise upper estimate (not much different from the conjectured precise value) or alternatively show indirectly that the mixing time is smaller than the hitting time of the loop-erased random walk.

APPENDIX A. PROOFS OF KNOWN AND UNSURPRISING FACTS

The harmonic potential on $\mathbb{Z}^d$, $d > 2$, is the unique bounded function a satisfying

$$\Delta a(z) = \begin{cases} 1 & z = \vec{0} \\ 0 & \text{otherwise} \end{cases}$$

and $a(\infty) = 0$ where Δ stands for the discrete Laplacian. It is well known (see e.g. [L96, theorem 1.5.4] [3] or [KS, theorem 5]) that $a(v) = \alpha|v|^{2-d} + O(|v|^{-d})$.

Lemma A.1. *Let $B_1 = B(x_1, r_1) \subset B_2 = B(x_2, r_2) \subset T_N^d$, $r_2 \leq C_1 r_1$. Let $v \in B_2 \setminus B_1$ satisfy $d(v, \partial B_2) \geq c_1 r_1$. Let R be a random walk starting from v and stopped on $\partial B_1 \cup \partial B_2$. Let p be the probability that R hits ∂B_1. Then $p \geq c(c_1, C_1)$.*

We assume here that a ball (e.g. B_2) satisfies $r_2 < \frac{1}{2}N$ i.e. it does not wrap itself because we are on a torus. This assumption holds for all balls in this appendix, and we will not repeat it.

Proof. Clearly, we may assume r_1 is sufficiently large in the sense that $r_1 > C(c_1, C_1)$. Since we are dealing with a process completely inside B_2, we may assume we are in $\mathbb{Z}^d$. Assume first that $|x_1 - v| < \frac{1}{2}d(x_1, \partial B_2)$. Since $a_1(v) := a(v - x_1)$ is harmonic on $B_2 \setminus B_1$, $a_1(R)$ is a martingale, and if we define τ to be the stopping time on $\partial B_1 \cup \partial B_2$ then we get $a_1(v) = \mathbb{E}a_1(R(\tau))$, so

$$a_1(v) = p\mathbb{E}(a_1(R(\tau)) \,|\, R(\tau) \in \partial B_1) + (1-p)\mathbb{E}(a_1(R(\tau)) \,|\, R(\tau) \in \partial B_2)$$
$$\leq p\alpha r_1^{2-d}(1 + o(1)) + (1-p)\alpha d(x_1, \partial B_2)^{2-d}(1 + o(1)) \tag{47}$$

(the $o(1)$ notations are as $r_1 \to \infty$ and may depend on c_1 and C_1) and from $a_1(v) = \alpha|x_1 - v|^{2-d}(1 + o(1))$ we get

$$p \geq \frac{|x_1 - v|^{2-d} - d(x_1, \partial B_2)^{2-d}}{r_1^{2-d} - d(x_1, \partial B_2)^{2-d}}(1 + o(1)) \geq c \tag{48}$$

for r_1 sufficiently large. In the case $|x_1 - v| \geq \frac{1}{2}d(x_1, \partial B_2)$, we can find a sequence of balls $B(y_n, s_n)$ of length $\leq C(c_1, C_1)$ and each $s_n \geq c(c_1, C_1)r_1$ such that $|y_1 - v| \leq \frac{1}{2}d(y_1, \partial B_2)$ and $\forall w \in \partial B(y_i, s_i)$, $|y_{i+1} - w| \leq \frac{1}{2}d(y_{i+1}, \partial B_2)$. Notice that this is possible because $d(v, \partial B_2) \geq c_1 r_1$. The previous case now gives that the probability that the random walk, after hitting $B(y_i, s_i)$ will continue to $B(y_{i+1}, s_{i+1})$ is $\geq c_2(c_1, C_1)$. Since it needs to perform only $C(c_1, C_1)$ such steps in order to hit B_1, we get $p \geq c_2^C = c_3(c_1, C_1)$. $\square$

Lemma A.2. *Let $B(x_1, r_1), B(x_2, r_2)$ be two balls, $r_2 \leq C_1 r_1$ and $|x_1 - x_2| \leq C_1 r_1$; and let $v \notin B_2 \cup B_1$ satisfy $d(v, B_2) \geq c_1 r_2$. Then $p \geq c(c_1, C_1)$ where p is as above.*

Proof. Assume first that $d(v, B_1) \leq \frac{1}{2}d(B_2, B_1)$ where $d(B_1, B_2)$ stands for the distance between the two balls in the usual sense. Let B_i' be the sets B_i considered as subsets of $\mathbb{Z}^d$ and let $S_i = B_i' + N\mathbb{Z}^d$ i.e. S_i is the preimage of B_i by the quotient map $\mathbb{Z}^d \to T_N^d$. Let R' be a simple random walk on $\mathbb{Z}^d$ starting from v (we consider v and the B_i's as subsets of $\mathbb{Z}^d$ as well, say by locating them in $[0, N]^d$). Then

$$p = \mathbb{P}(R' \text{ hits } S_1 \text{ before } S_2) \geq \mathbb{P}(R' \text{ hits } B_1 \text{ before } S_2) \geq$$

$$\geq \mathbb{P}(R' \text{ hits } B_1 \text{ before } \partial B(x_1, r_1 + d(B_1, B_2))) \overset{(*)}{\geq} c$$

---

[3] [L96] only shows $a(v) = \alpha|v|^{2-d} + O(|v|^{\epsilon - d})$, but this is completely sufficient for our purposes.

where $(*)$ comes from the same harmonic potential arguments as (47)-(48).

If $d(v, B_1) > \frac{1}{2}d(B_2, B_1)$ but we have both

$$d(v, B_1) \leq (2C_1 + 2)r_1$$
$$d(v, B_2) \geq c_1 r_1$$

then the same ball-sequence argument as in the previous lemma gives $p \geq c$;

If $d(v, B_1) > (2C_1 + 2)r_1$, let τ be the hitting time of $B_3 := B(x_1, (2C_1 + 2)r_1)$, then

$$p = \mathbb{E}\mathbb{P}(R' \text{ starting from } R(\tau) \text{ hits } B_1 \text{ before } B_2) \geq \mathbb{E}c = c \qquad (49)$$

where here R' is a simple random walk on T_N^d (differing from R only by the starting point), the expectation $\mathbb{E}$ is over the distribution of $R(\tau)$ and the inequality comes from the previous two cases.

Finally, if $d(v, B_2) < c_1 r_1$ define τ the hitting time of $B_4 := B(x_2, c_1 r_1)$. The harmonic potential at x_2 with a calculation similar to (47)-(48) shows that the probability to hit B_4 before B_2 is $\geq c$. After hitting B_4 a calculation similar to (49) gives that $p \geq c$. $\qquad \square$

Lemma A.3. *Let $d(B(x_1, r_1), B(x_2, r_2)) \geq c_1 r_1$, $r_2 \geq c_1 r_1$ and $|x_1 - x_2| \leq C_1 r_2$; and let $v \in \partial B_1$. Let R be a random walk starting from v and stopped on $\partial B_2 \cup \{x_1\}$. Let p be the probability that R hits x_1. Then $p \approx r_1^{2-d}$.*

In the formulation of the lemma, and in its proof, all constants implicit in the $\approx$ signs might depend on c_1 and C_1.

Proof. Let $B_3 = B(x_1, \frac{1}{2}r_1)$. Define stopping times t_i similarly to (2), as follows: $t_0 := 0$ and

$$t_{2i+1} := \{t > t_{2i} : R(t) \in \partial B_3)$$
$$t_{2i} := \{t > t_{2i-1} : R(t) \in \partial B_1 \cup \{x_1\}\}.$$

Define also τ the hitting time of ∂B_2. The usual harmonic potential calculations (use the harmonic potential around x_1) show that the probability of a random walk starting from any $v \in \partial B_3$ to hit x_1 before exiting B_1 is $\approx r_1^{2-d}$. Hence, $\mathbb{P}(R(t_{2i}) = x_1 | R(t_{2i-1})) \leq Cr_1^{2-d}$. Lemma A.2 shows that a random walk starting from any point in ∂B_1 has probability $\geq c$ to hit B_2 before hitting B_3. Therefore, the probability to get $t_i > \tau$ decreases exponentially in i i.e. $\mathbb{P}(t_{2i} < \tau) \leq Ce^{-ci}$ and hence

$$\mathbb{P}(R(t_{2i}) = x_1 \text{ and } t_{2i} < \tau) \leq Ce^{-ci}r_1^{2-d}$$

so

$$p \leq \sum_{i=0}^{\infty} \mathbb{P}(R(t_{2i+1}) = x_1 \text{ and } t_{2i+1} < \tau) \leq Cr_1^{2-d}.$$

The inequality $p \geq cr_1^{2-d}$ follows easily from (47)-(48) for the harmonic potential at x_1 and the requirement $d(B_1, B_2) \geq c_1 r_1$. $\qquad \square$

Lemma A.4. *Let $|v| < (1-c_1)r$ and $w \in \partial B(0, r)$. Then the probability p that a random walk starting from v will exit B in w is $\approx r^{1-d}$. Without the restriction $|v| < (1 - c_1)r$ one has $p \leq C(r - |v|)^{1-d}$*

In the formulation of the lemma, and in its proof, all constants implicit in the $\approx$ signs might depend on c_1.

Proof. In $\mathbb{Z}^d$, $d > 2$, the probability of a walk starting from v to never return is $> c$. Hence the probability to hit $\partial B(0, r)$ before returning to v is $> c$, and this event is identical on $\mathbb{Z}^d$ and on the torus. The symmetry of the random walk shows that p is $\approx$ to the probability that a random walk starting from w will hit $\partial B \cup \{v\}$ in v (the quotient is exactly the probability of a random walk starting from v to return to v before exiting ∂B, which, as we just discussed, is ≈ 1)[4]. This probability can be calculated in three steps as follows. First, the probability of a random walk starting from w to hit $\partial B(0, \frac{2}{3}r + \frac{1}{3}|v|)$ is $\approx (r - |v|)^{-1}$: this uses an argument similar (47)-(48) using the harmonic potential a at 0, but here we need the precise estimate $a(x) = |x|^{2-d} + O(|x|^{-d})$ or at least $a(x) = |x|^{2-d} + O(|x|^{1-d})$ (see, e.g. [K87, lemma 3] for a detailed version of this calculation). Next, if $|v| < r(1 - c_1)$, use lemma A.1 to show that continuing from any point on $\partial B(0, \frac{2}{3}r + \frac{1}{3}|v|)$ the probability to hit $B(v, \frac{1}{3}(r - |v|))$ is ≈ 1 — if $|v| \geq r(1 - c_1)$, we only estimate that this probability is ≤ 1. Finally, the same (47)-(48) argument with the harmonic potential at v shows that starting from any point on $\partial B(v, \frac{1}{3}(r - |v|))$, the probability to hit v before hitting ∂B is $\approx (r - |v|)^{2-d}$. $\qquad\square$

A similar calculation works when $(1 + c_1)r \leq ||v||$ and p is the probability the random walk will hit B in w, using lemma A.2 instead of lemma A.1 and lemma A.3 in the third step.

Lemma A.5. *Let $v \in B(0, r) \subset B(0, 2r) \subset T_N^d$. Let R be a random walk starting from v and stopped on $\partial B(0, 2r)$. Let R_x be a random walk starting from v and conditioned to hit $\partial B(0, 2r)$ at a specific point x. Then $R \cap B(0, r) \approx R_x \cap B(0, r)$ where $\approx$ means that the probabilities of any event are equal up to a constant.*

Proof. Let t be the last time when $R(t) \in B(0, r)$. Let $w = R(t)$. For any w, the probability of an (unconditioned) random walk starting from w to hit $B(0, r) \cup \partial B(0, 2r)$ in $\partial B(0, 2r)$ is $\approx r^{-1}$. The probability to hit x is $\approx r^{-d}$. This independence from w finishes the lemma. Both estimates are easily proved as in the previous lemma. $\quad\square$

Lemma A.6. *Let R be a random walk starting from $v \in B(0, r)$ and let $w \in B(0, r)$. Let $t > r^2$. Let p be the probability that $R[0, t] \subset B(0, r)$ and $R(t) = w$. Then*

$$p \leq Cr^{-d}e^{-ctr^{-2}} \quad.$$

Proof. Starting from any $v \in B(0, r)$, after r^2 steps the random walk has probability $> c$ to exit $B(0, r)$. This shows, clearly, that the probability that $R[0, t - r^2] \subset B(0, r)$ is $\leq Ce^{-ctr^{-2}}$. For any $x \in B(0, r)$, the probability that a random walk R' starting from x satisfies $R'(r^2) = w$ is $\leq Cr^{-d}$. $\qquad\square$

Lemma A.7. *Let $v \in B(0, r)$ and let R be a random walk starting from v and stopped on $\partial B(0, r)$. Let $w \in \partial B(0, r)$. Then the probability p that R hits w satisfies*

$$p \leq C_1|v - w|^{1-d} \quad.$$

Proof. Denote $s = |v - w|$. We shall prove the lemma using an induction process that assumes the lemma holds for $1, \ldots, \frac{1}{2}s$ and proves it for $\frac{1}{2}s + 1, \ldots, s$.

Denote $d(v, \partial B(0, r)) = \epsilon s$. The first thing to note is that the lemma holds if $\epsilon > c$ with no need for induction (in the sense that $p \leq C_2(c)|v - w|^{1-d}$), due to the second part of lemma A.4.

---

[4]When we say "hit" we mean at time > 0, so that these probabilities are not simply 1.

It is for the case of small ϵ that we need the induction process. Let $\delta = c_1 \log^{-1} \epsilon^{-1}$ for some c_1 which will be fixed later. Let

$$D := B(0,r) \setminus B(0, r - \delta s)$$
$$E := D \cap B(v, \tfrac{1}{2}s) \quad .$$

Examine the exit probabilities of R from E. Let p_2 be the probability that R exits E at $\partial B(0, r - \delta s) \cap \partial E$. Then

$$p_2 \leq \mathbb{P}(R \text{ exits } D \text{ at } \partial B(0, r - \delta s)) \leq C\epsilon/\delta$$

where the second inequality comes from the harmonic potential at zero. Let p_3 be the probability that R exits E at $\partial B(v, \tfrac{1}{2}s) \cap \partial E$. It is easy to see that for any $x \in D$, the probability that R exits $B(x, 2\delta s)$ without hitting ∂D is $< 1 - c$. Therefore

$$p_3 < (1 - c)^{(s/2)/(2\delta s)} = e^{-cc_1^{-1} \log \epsilon^{-1}} \quad .$$

Therefore for c_1 sufficiently small we would get $p_3 \leq \epsilon$. Together these two give

$$p_2 + p_3 \leq C\epsilon \log \epsilon^{-1} \quad .$$

Since $E \cap B(w, \tfrac{1}{2}s) = \emptyset$ we get that the probability of R to hit $\partial B(w, \tfrac{1}{2}s)$ before exiting $B(0,r)$ is $\leq C\epsilon \log \epsilon^{-1}$. The induction assumption gives that the probability to hit w after hitting $\partial B(w, \tfrac{1}{2}s)$ is $\leq C_1(\tfrac{1}{2}s)^{1-d}$. Therefore

$$p \leq C_1 s^{1-d} \cdot (C2^{d-1}\epsilon \log \epsilon^{-1}) \quad .$$

For $\epsilon < c_2$ this will be $\leq C_1 s^{1-d}$ and this case is finished too. The lemma is now finished because for $\epsilon < c_2$ the induction process works for any C_1, and for $\epsilon \geq c_2$ the first case allows to define $C_1 := C_2(c_2)$. $\square$

A similar calculation works when v is outside $B(0,r)$, and the random walk is stopped when hitting $B(0,r)$ and the conclusion is $p \leq C \min\{|v-w|, r\}^{1-d}$.

Lemma A.8. *Let $v \in B(0, \tfrac{1}{2}r)$ and $w \in \partial B(0,r)$. Let R be a random walk started from v and conditioned to exit $B(0,r)$ in w. Let t be the exit time. Then*

$$\mathbb{P}(t > \lambda r^2) \leq Ce^{-c\lambda} \quad .$$

Proof. We may assume $\lambda > 1$. Let R' be an unconditioned walk starting from v. Let

$$A_n = (B(w, 2^n) \setminus B(w, 2^{n-1})) \cap B(0,r) \quad .$$

Lemma A.6 shows that

$$\mathbb{P}(R'[0, \lambda r^2] \subset B(0,r) \text{ and } R'(\lambda r^2) \in A_n) \leq C\#A_n r^{-d} e^{-c\lambda} \quad .$$

Lemma A.7 shows that for every $x \in A_n$, the probability of a random walk starting at x to exit $B(0,r)$ at w is $\leq C2^{n(1-d)}$. Therefore by lemma A.8

$$\mathbb{P}(R'[0, \lambda r^2] \subset B(0,r) \text{ and } R'(\lambda r^2) \in A_n \text{ and } R' \text{ hits } w) \leq$$
$$\leq C(\#A_n)2^{n(1-d)}r^{-d}e^{-c\lambda} \leq Cr^{-d}2^n e^{-c\lambda}$$

and we get

$$\mathbb{P}(R'[0, \lambda r^2] \subset B(0,r) \text{ and } R' \text{ hits } w) \leq Cr^{1-d}e^{-c\lambda} \sum_{n=1}^{\lfloor \log r \rfloor} r^{-1}2^n \leq Cr^{1-d}e^{-c\lambda} \quad .$$

Since the probability of R' to hit w is $> cr^{1-d}$ (by lemma A.4), we are done. $\square$

Lemma A.9. *Let X_i be events with a past-independent exponential estimate, namely*

$$\mathbb{P}(X_i > \lambda E \,|\, X_{i-1}, \ldots, X_1) \leq C_1 e^{-c_1 \lambda}.$$

Then

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i > \lambda n E\right) \leq C e^{-c\lambda}.$$

As usual, C, c and all constants in the proof might depend on C_1 and c_1.

Proof. Clearly we may assume $E = 1$. Let Y_i be i.i.d variables with $Y_i \sim C_2(1+G)$ where G is a standard exponential variable (namely with density e^{-t}) and C_2 is some constant sufficiently large such that $\mathbb{P}(Y_i > \lambda) \geq \min\{1, C_1 e^{-c_1\lambda}\}$. A simple induction now shows that

$$\mathbb{P}\left(\sum X_i > \lambda n\right) \leq \mathbb{P}\left(\sum Y_i > \lambda n\right)$$

and the sum of the Y_i has the distribution $C_2(n+\Gamma)$ where Γ has density $e^{-t}\frac{t^{n-1}}{(n-1)!}$. A simple calculation shows that $\mathbb{P}(\Gamma > \lambda n) \leq C e^{-c\lambda}$. $\qquad\square$

Lemma A.10. *Let $v \in T_N^d$, and let R be a random walk starting from v going to a length of $C_1 N^d r^{2-d}$ for some C_1 sufficiently large. Then*

$$\mathbb{P}(R \text{ hits } B(0,r)) \geq \frac{1}{2} \quad \forall v \in T_N^d.$$

Proof. Define S to be the preimage of B in $\mathbb{Z}^d$ (namely $B + N\mathbb{Z}^d$) and let R' be a random walk of on $\mathbb{Z}^d$ starting from some preimage of v. Then

$$\mathbb{P}(R \text{ hits } B(0,r)) = \mathbb{P}(R'[0, N^d r^{2-d}] \cap S \neq \emptyset).$$

Define stopping times t_i as follows: $t_0 = 0$ and for every i let z_i be the element of $N\mathbb{Z}^d$ closest to $R(t_i)$. Define inductively

$$t_{i+1} = \min\{t > t_i : R(t) \in \partial B(z_i, r) \cup \partial B(z_i, 2N)\}.$$

Since $d(R(t_i), z_i) \leq N$ then using the harmonic potential at z_i shows that

$$\mathbb{P}(R(t_{i+1}) \in S) \geq \mathbb{P}(R(t_{i+1}) \in \partial B(z_i, r) \geq c(r/N)^{d-2}$$

independently of the value of $R(t_i)$. This immediately gives that, for C_2 sufficiently large

$$\mathbb{P}(R[0, t_{C_2(N/r)^{d-2}}] \cap S = \emptyset) \leq \left(1 - c\left(\frac{r}{N}\right)^{d-2}\right)^{C_2(N/r)^{d-2}} \leq \frac{1}{4}. \qquad (50)$$

On the other hand, it is easy to see that $\mathbb{P}(t_{i+1} - t_i > \lambda N^2) \leq C e^{-c\lambda}$ independently of the past, and using lemma A.9 we get that $\mathbb{P}(t_n > \lambda n N^2) \leq C e^{-c\lambda}$. Using this for $n = C_2(N/r)^{d-2}$ and λ sufficiently large we get that

$$\mathbb{P}(t_n > C N^d r^{2-d}) \leq \frac{1}{4}. \qquad (51)$$

(50) and (51) together show that the C in (51) may serve as our C_1. $\qquad\square$

Lemma A.11. *Let $v \in \partial B(0, 2r)$ and $w \in \partial B(0, r)$. Let R be a random walk started from v and conditioned to hit $B(0,r)$ in w. Let t be the hitting time. Then*

$$\mathbb{P}(t > \lambda N^d r^{2-d}) \leq C e^{-c\lambda} \quad .$$

The proof is identical to that of lemma A.8 with the use of lemmas A.4 and A.7 replaced by the comments following them, respectively, and using lemma A.10 to show that the probability to not hit a ball of radius r after $\lambda N^d r^{2-d}$ steps is $\leq Ce^{-c\lambda}$ and hence the equivalent of lemma A.6. We omit the details.

References

[BKPS]   Itai Benjamini, Harry Kesten, Yuval Peres and Oded Schramm, *Geometry of the Uniform Spanning Forest: Transitions in Dimensions 4, 8, 12*, to appear in Ann. of Math, http://arXiv.org/abs/math.PR/0107140

[BLPS01] Itai Benjamini, Russell Lyons, Yuval Peres and Oded Schramm, *Uniform spanning forests*, Ann. Probab. **29:1** (2001), 1–65.

[DS98]   Eric Debez and Gordon Slade, *The scaling limit of lattice trees in high dimensions*, Communications in Mathematical Physics **193:1** (1998), 69–104.

[GB90]   A. J. Guttmann and R. J. Bursill, *Critical exponents for the loop erased self-avoiding walk by Monte Carlo methods*, Journal of Statistical Physics, **59:1/2** (1990), 1–9.

[K00a]   Rick Kenyon, *The asymptotic distribution of the discrete Laplacian*, Acta Mathematica **185:2** (2000), 239–286.

[K00b]   Rick Kenyon, *Long range properties of spanning trees*, J. Math. Phys., **41:3** (2000) 1338–1363.

[HS90]   Takashi Hara and Gordon Slade, *Mean-field critical behavior for percolation in high dimensions*, Communications in Mathematical Physics **128** (1990), 333–391.

[HS92]   Takashi Hara and Gordon Slade, *Self-avoiding walk in five or more dimensions. I. The critical behaviour*, Communications in Mathematical Physics, **147:1** (1992), 101–136.

[K87]   Harry Kesten, *Hitting probabilities of random walks on $\mathbb{Z}^d$*, Stochastic Processes and their Applications **25** (1987), 165–184.

[K]   Gady Kozma, *Scaling limit of loop erased random walk — a naive approach*, http://arXiv.org/abs/math.PR/0212338

[KS]   Gady Kozma and Ehud Schreiber, *An asymptotic expansion for the discrete harmonic potential*, http://arxiv.org/abs/math.PR/0212156

[L80]   Gregory F. Lawler, *A self-avoiding random walk*, Duke Math. J. **47:3** (1980), 655–693.

[L87]   Gregory F. Lawler, *Loop-erased self-avoiding random walk and the Laplacian random walk*, J. Phys. A **20:13** (1987), 4565–8.

[L95]   Gregory F. Lawler, *The logarithmic correction for loop-erased walk in four dimensions*, Proceedings of the conference in honor of Jean-Pierre Kahane (Orsay, 1993), special issue of J. Fourier Anal. Appl. (1995), 347–362.

[L96]   Gregory F. Lawler, *Intersections of random walks*, Birkhaüser Boston, 1996.

[L99]   Gregory F. Lawler, *Loop-erased random walk*, in *Perplexing problems in probability*, Birkhäuser Boston, 1999, 197–217.

[LSW]   Gregory F. Lawler, Oded Schramm and Wendelin Werner, *Conformal invariance of planar loop-erased random walk and uniform spanning trees*, to appear in Ann. Prob., http://arXiv.org/abs/math.PR/0112234

[LEP86]   J. W. Lyklema, C. Evertz and L. Pietronero, *The Laplacian random walk*, Europhysics-Letters **2:2** (1986), 77–82.

[LPS]   Russell Lyons, Yuval Peres and Oded Schramm, *Markov Chain Intersections and the Loop-Erased Walk*, To appear in Ann. Inst. H. Poincaré Probab. Statist., http://arxiv.org/abs/math.PR/0107055

[P91]   Robin Pemantle. *Choosing a spanning tree for the integer lattice uniformly*. Ann. Probab., **19:4** (1991), 1559–1574.

[S00]   Oded Schramm, *Scaling limits of random walks*, Israel Journal of Mathematics, vol. **118** (2000), 221–288.

[S95]   Gordon Slade, *The critical behavior of random systems*. Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 1315–1324, Birkhäuser Basel, 1995.

[W96]   David Wilson, *Generating random spanning trees more quickly than the cover time*, Twenty-Eighth Annual ACM symposium on Theory of Computing, 293–303, http://research.microsoft.com/~dbwilson/ja/tau.ps